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ON THE RESOLUTION OF HIGHER SINGULARITIES OF ALGEBRAIC CURVES INTO ORDINARY NODES

BY

B. M. WALKER

A DISSERTATION

SUBMITTED TO THE FACULTIES OF THE GRADUATE SCHOOLS OF ARTS, LITERATURE, AND SCIENCE IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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ON THE RESOLUTION OF HIGHER SINGULARITIES OF ALGEBRAIC CURVES INTO ORDINARY NODES.

Introduction.

Every algebraic curve C can be transformed by a birational transformation *of the plane* (a so-called Cremona transformation) into a curve C' which has no other singular points than ordinary * multiple points. This fundamental theorem was established by M. Noether † in 1875.

By a birational transformation *of the curve* (a so-called Riemann transformation), the curve C' can further be transformed into a curve C'' which has *no other singular points than ordinary * double points.* It seems that this theorem was first enunciated by Halphen.‡

Three essentially different methods have been employed for its proof:

1) Picard § uses a quadratic one-to-three transformation of the plane.

2) Bertini || uses, for the same purpose, a cubic one-to-two transformation of the plane.

* By an *ordinary multiple* point, we mean, in the sequel, a multiple point all of whose cycles are linear and have distinct tangents. For the definition of cycles, see Halphen, "Traité de Géométrie de Salmon," p. 541, and Jordan, "Cours D'Analyse," t. I, p. 563.

† Noether, "Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen curve," *Mathematische Annalen*, Bd. 9, p. 166. Compare also, Picard, "Traité D'Analyse," t. II, p. 364, Jordan, "Cours D'Analyse," t. I. p. 588, and Halphen, "Traité de Géométrie de Salmon," p. 630.

‡ Halphen, "Traité de Géométrie de Salmon," pp. 630, 631, 632. His proof is, however, incomplete, as has been pointed out by Picard, "Traité D'Analyse," t. II, p. 366.

§ Picard, "Traité D'Analyse," t. II, p. 366, 1893, and Simart, "Sur un théorème relatif à la transformation des courbes algébriques," *Comptes rendus de l'Académie*, t. 116, p. 1047, 1893.

|| Bertini, "Trasformazióne di una curva algebrica in un'altra con soli punti doppi," *Mathematische Annalen*, Bd. 44, p. 158, 1894.

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3) POINCARÉ * transforms the plane curve into a curve in space, which has no singular points; and then shows that this curve may be projected into a plane curve with no other singular points than ordinary double points. All these proofs, however, are written in an exceedingly concise style and leave a great many minor points to the reader.

The object of the present paper is to give a new proof of the theorem in question by carrying out, in detail, an idea contained in the following foot-note added by Professor *Felix Klein* to Bertini's paper:

"Die Methode von Bertini kommt geometrisch zu reden darauf zurück, die Ebene, in welcher uns die Curve mit singulären Punkte gegeben ist, als eindeutige Abbildung einer Fläche 3. Ordnung zu betrachten, dadurch die Curve in eine Raumcurve zu verwandeln und letztere hinterher wieder von einen hinreichend allgemeinen Punkte aus auf eine andere Ebene zu projiciren. In dieser Form ist mir der Ansatz noch von *Clebsch* her bekannt, der mir denselben in Herbst 1869 mündlich mittheilte."

The paper is divided into four chapters. In Chapter I, those properties of triply infinite systems of cubics are established, which are needed for the transformation from the plane to the cubic surface. Chapters II and III deal with the transformation from the plane to the cubic surface, † and its effect upon the given algebraic curve. Let $A_1, A_2, \cdots, A_r$ be the singular points of the given algebraic curve K in the original plane Π; they are all supposed to be ordinary multiple points. It is proved that the six fundamental points of the transformation can always be so chosen that one of the multiple points of K, say A_1, is resolved into simple points of the image K' of K on the cubic surface and all the remaining multiple points $A_2, \cdots, A_r$ of K are transformed into new ordinary ‡ multiple points

* POINCARÉ, "Sur les transformations birationelles des courbes algébriques," *Comptes rendus de l'Académie*, t. 117, p. 18, 1893. For additional literature see "Encyklopädie der Mathematischen Wissenschaften," Band III_2, Heft 3, 1906.

† Clebsch's well-known one-to-one correspondence between plane and cubic surface ("Geometrie auf den Flächen dritter Ordnung," *Crelle*, 65).

‡ Each one of these points is the center of a finite number of linear cycles (JORDAN, "Cours D'Analyse," t. I, p. 563) with distinct tangents.

$A_2', \cdots, A_r'$ of K' unchanged in orders of multiplicity and with distinct tangents.

In Chapter IV, the curve K' is projected into a new curve K'', in a plane Π', from a center of projection O chosen on the cubic surface. It is proved that the point O can always be so chosen that the curve K'' possesses one multiple point less than the curve K and has all the remaining multiple points $A_2, \cdots, A_r$ of K transformed into new ordinary multiple points $A_2'', \cdots, A_r''$ unchanged in orders of multiplicity and with distinct tangents, and a finite number of additional ordinary double points, but no other singular points.

A finite number of repetitions of this process leads to a curve which has no other singular points than ordinary nodes.

CHAPTER I.

Properties of Triply Infinite Linear Systems of Plane Cubics.

§ 1. Triply infinite linear systems of cubics.

We shall develop, in this chapter, a number of auxiliary theorems on triply infinite linear systems of cubics, which will be needed in our discussion of the one-to-one correspondence between the plane and the cubic surface.

In a plane Π, whose points are referred to a system of trilinear coordinates $x : y : z$, there are given six distinct points $P_i(x_i, y_i, z_i)$, $(i = 1, 2, \cdots, 6)$, chosen so that *no three points lie on a straight line.* We consider the problem:

To determine all cubics passing through the six points.

Let

$$f \equiv Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gxz^2 + Hy^2z + Iyz^2 + Jxyz = 0, \tag{1}$$

where $A, B, C, \cdots$, are arbitrary constants and x, y, z are homogeneous coördinates, be the general equation of a cubic.

If the cubic, $f = 0$, is to pass through the six points P_i, the coefficients $A, B, \cdots$, must satisfy the six equations

$$Ax_i^3 + By_i^3 + Cz_i^3 + \cdots = 0 \qquad (i = 1, 2, \cdots, 6). \tag{2}$$

The matrix of the coefficients,

$$\mathfrak{M} \equiv \| x_i^3 \quad y_i^3 \quad z_i^3 \quad x_i^2y_i \quad x_i^2z_i \quad x_iy_i^2 \quad x_iz_i^2 \quad y_i^2z_i \quad y_iz_i^2 \quad x_iy_iz_i \|,$$

is always of rank six; that is, at least, one of its determinants of order six is $\neq 0$. For, if we choose the vertices of the triangle of reference to coincide with the points P_1, P_2, P_3, respectively, we have

$$y_1 = z_1 = 0, \; x_2 = z_2 = 0, \; x_3 = y_3 = 0,$$

and therefore, since no three of the six points are to lie on a straight

line, $x_4, \cdots, z_6 \neq 0$; moreover, since P_4, P_5, P_6 do not lie on a straight line,

$$\begin{vmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{vmatrix} \neq 0.$$

Therefore the determinant

$$\begin{vmatrix} x_4^2 y_4 & x_4 y_4^2 & x_4 y_4 z_4 \\ x_5^2 y_5 & x_5 y_5^2 & x_5 y_5 z_5 \\ x_6^2 y_6 & x_6 y_6^2 & x_6 y_6 z_6 \end{vmatrix},$$

to which, in the present case, one of the determinants of $\mathfrak{M}$ reduces, is $\neq 0$, and consequently the rank of $\mathfrak{M}$, which is invariant under a transformation of coördinates, is *six*. At the same time, we obtain from (2), for this special triangle of reference, $A = B = C = 0$; and if we solve equations

$$Dx_i^2 y_i + Ex_i^2 z_i + Fx_i y_i^2 + \cdots = 0 \qquad (i = 4, 5, 6), \tag{3}$$

with respect to D, F, J and substitute the values in (1), we get, for the system of cubics,

$$\begin{aligned} &H(\alpha_1' x^2 + \beta_1' xy + \gamma_1' xz + yz)y + G(\alpha_2' xy + \beta_2' y^2 \\ &\quad + \gamma_2' yz + z^2)x + I(\alpha_3' x^2 + \beta_3' xy + \gamma_3' xz + z^2)y \\ &\quad + E(\alpha_4' xy + \beta_4' y^2 + \gamma_4' yz + xz)x = 0, \end{aligned} \tag{4}$$

where α_1', β_1', $\cdots$, are fixed constants. Hence, if we denote

$$\begin{cases} f_1 \equiv (\alpha_1' x^2 + \beta_1' xy + \gamma_1' xz + yz)y \equiv \phi_1 y, \\ f_2 \equiv (\alpha_2' xy + \beta_2' y^2 + \gamma_2' yz + z^2)x \equiv \phi_2 x, \\ f_3 \equiv (\alpha_3' x^2 + \beta_3' xy + \gamma_3' xz + z^2)y \equiv \phi_3 y, \\ f_4 \equiv (\alpha_4' xy + \beta_4' y^2 + \gamma_4' yz + xz)x \equiv \phi_4 x, \end{cases} \tag{5}$$

$\alpha = H$, $\beta = G$, $\gamma = I$, $\delta = E$; we obtain the totality of cubics

through the six points in the form

$$\Sigma \equiv \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4 = 0, \tag{6}$$

$\alpha, \beta, \gamma, \delta$ being arbitrary constants. The four cubics f_1, f_2, f_3, f_4 are linearly independent; that is, it is impossible to determine four constants $\alpha, \beta, \gamma, \delta$, not all zero, so that

$$\alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4 \equiv 0, \tag{7}$$

since each one of these four cubics f_1, f_2, f_3, f_4 contains a term which does not occur in the other three; namely, the terms y^2z, xz^2, yz^2, x^2z respectively. Hence *the cubics through six points form*, WITHOUT EXCEPTION, *a triply infinite system*, PROVIDED THAT NO THREE OF THE SIX POINTS LIE ON A STRAIGHT LINE.

§ 2. Reduction of the base cubics to canonical form.

Instead of the four base cubics f_1, f_2, f_3, f_4, we introduce four other base cubics each of which breaks up into three straight lines, in the following manner: f_1 and f_3 have the common factor y, and the two conics, $\phi_1 = 0$, $\phi_3 = 0$, pass through the four fundamental points P_2, P_4, P_5, P_6; hence the pencil, $\alpha\phi_1 + \gamma\phi_3 = 0$, contains the product of the two lines, P_4P_6, P_2P_5, say for $\alpha = \alpha'$, $\gamma = \gamma'$, and also the product of the lines, P_4P_5, P_2P_6, say for $\alpha = \alpha''$, $\gamma = \gamma''$. Hence the two cubics,

$$\begin{cases} \bar{f}_1 \equiv \alpha' f_1 + \gamma' f_3, \\ \bar{f}_3 \equiv \alpha'' f_1 + \gamma'' f_3, \end{cases} \tag{8}$$

break up into the lines $P_1P_3 \cdot P_4P_6 \cdot P_2P_5$ and $P_1P_3 \cdot P_4P_5 \cdot P_2P_6$ respectively. Similarly the constants β', δ' and β'', δ'' can be so selected that the two cubics,

$$\begin{cases} \bar{f}_2 \equiv \beta' f_2 + \delta' f_4, \\ \bar{f}_4 \equiv \beta'' f_2 + \delta'' f_4, \end{cases} \tag{9}$$

break up into the lines $P_2P_3 \cdot P_4P_6 \cdot P_1P_5$ and $P_2P_3 \cdot P_1P_6 \cdot P_4P_5$ respectively.

Since $\alpha'\gamma'' - \gamma'\alpha''$ and $\beta'\delta'' - \beta''\delta'$ are evidently $\neq 0$, the four new cubics, $\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4$, are again independent and can therefore be used for base cubics.

In order to obtain the expressions of the cubics, $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, $\bar{f}_4$, in terms of x, y, z, let the equations of the lines, P_5P_6, P_4P_6, P_4P_5, be respectively,

$$(10) \qquad \begin{cases} \lambda \equiv a_1x + a_2y + a_3z = 0, \\ \mu \equiv b_1x + b_2y + b_3z = 0, \\ \nu \equiv c_1x + c_2y + c_3z = 0, \end{cases}$$

where $a_1, \cdots, c_3$ are constants.

Since the points, P_4, P_5, P_6, are distinct and no three of the points, P_i $(i = 1, 2, \cdots, 6)$, shall lie on the same straight line, the coefficients, $a_1, \cdots, c_3$, must satisfy the following conditions:

1) $$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0;$$

2) all the first minors of $\triangle \neq 0$, since

$$x_4 : y_4 : z_4 = b_2c_3 - b_3c_2 : b_3c_1 - b_1c_3 : b_1c_2 - b_2c_1, \text{ etc.};$$

if, for instance, $b_2c_3 - b_3c_2 = 0$, the points, P_2, P_3, P_4, would lie on the same line, etc.;

3) all the elements of $\triangle \neq 0$, for if, for instance, $a_1 = 0$, the line P_5P_6 would pass through the point P_1, etc.

Calling the new base cubics again f_1, f_2, f_3, f_4, their analytic expressions are

$$(11) \qquad \begin{cases} f_1 \equiv y\phi_1 \equiv \mu y(c_2\lambda - a_2\nu) \equiv \mu y(\beta_1 z - \beta_3 x) \equiv P_4P_6 \cdot P_1P_3 \cdot P_2P_5, \\ f_2 \equiv x\phi_2 \equiv \mu x(c_1\lambda - a_1\nu) \equiv \mu x(\beta_3 y - \beta_2 z) \equiv P_4P_6 \cdot P_2P_3 \cdot P_1P_5, \end{cases}$$

$$(12) \qquad \begin{cases} f_3 \equiv y\phi_3 \equiv \nu y(b_2\lambda - a_2\mu) \equiv \nu y(\gamma_3 x - \gamma_1 z) \equiv P_4P_5 \cdot P_1P_3 \cdot P_2P_6, \\ f_4 \equiv x\phi_4 \equiv \nu x(b_1\lambda - a_1\mu) \equiv \nu x(\gamma_2 z - \gamma_3 y) \equiv P_4P_5 \cdot P_2P_3 \cdot P_1P_6, \end{cases}$$

where $\alpha_1, \cdots, \gamma_3$ are the minors of $\triangle$ corresponding to the elements $a_1, \cdots, c_3$ respectively. From the preceding normal form, the following results are easily proved:

1) The base cubics have no other common point of intersection except the six points $P_i\,(i = 1, 2, \cdots, 6)$.

2) Not one of the points, P_i, is a multiple point on all the base cubics.

3) At none of the points, P_i, have *all* the base cubics coinciding tangents.

(Compare the adjoining diagram.)

§ 3. Rank of the matrix of the first derivatives of f_1, f_2, f_3, f_4.

We consider, in this section, the matrix

$$\left\| \begin{matrix} f_{1x} & f_{2x} & f_{3x} & f_{4x} \\ f_{1y} & f_{2y} & f_{3y} & f_{4y} \\ f_{1z} & f_{2z} & f_{3z} & f_{4z} \end{matrix} \right\|, \tag{13}$$

where

$$f_{ix} \equiv \frac{\partial f_i}{\partial x}, \qquad f_{iy} \equiv \frac{\partial f_i}{\partial y}, \qquad f_{iz} \equiv \frac{\partial f_i}{\partial z}; \tag{14}$$

and propose to determine all points in the plane for which the rank, r, of the matrix is 0, 1, 2 respectively.

1° $\qquad r = 0.$

When $r = 0$, all the elements in the matrix vanish; therefore, since

$$x f_{ix} + y f_{iy} + z f_{iz} \equiv 3 f_i,$$

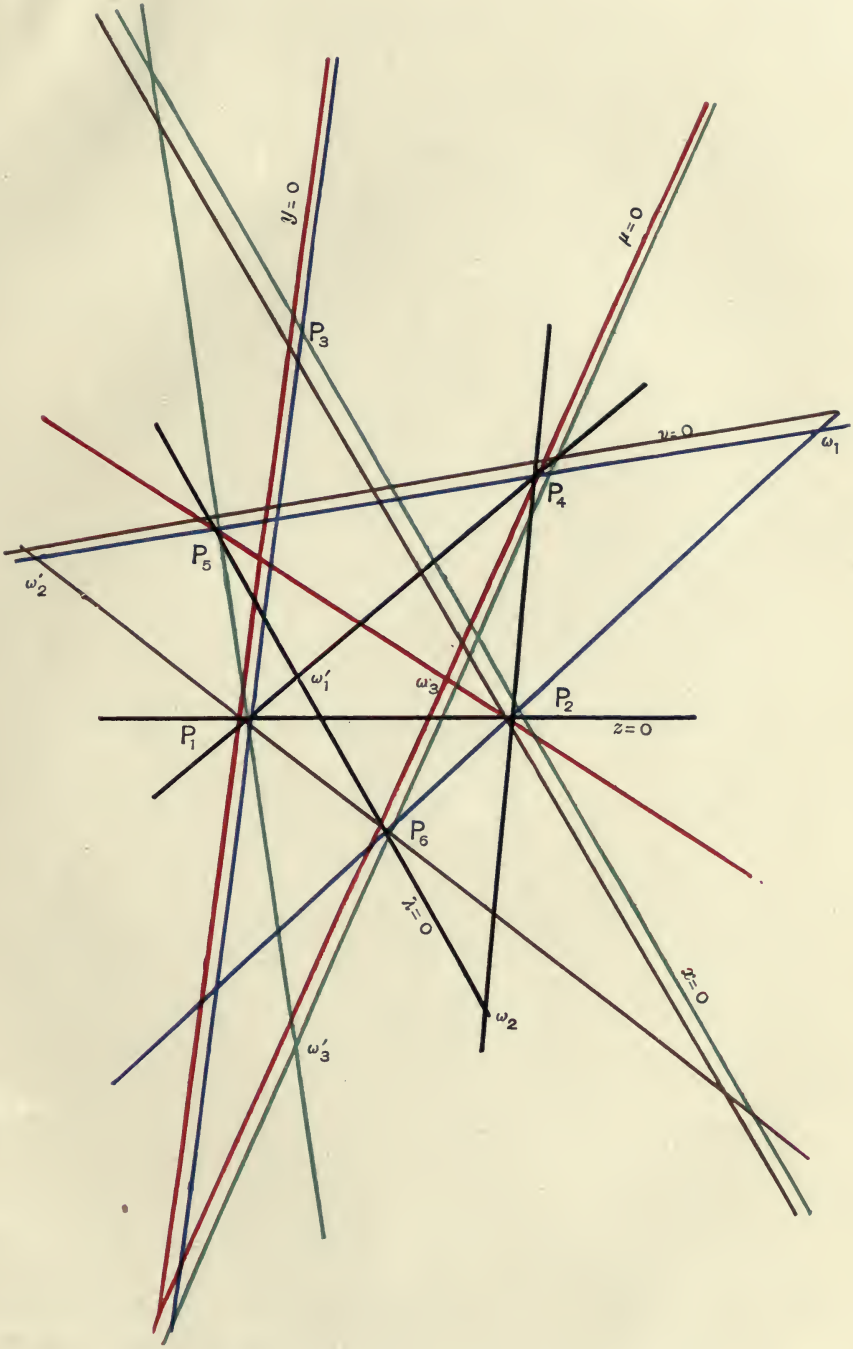
the point is a common point of intersection and moreover a double point on each base cubic, against the result already established.

Therefore, *there are no points in the plane for which* $r = 0$.

2° $\qquad r = 1.$

When $r = 1$, all the determinants of order *two* vanish; whereas, at least, one element is $\neq 0$. Hence it follows that at a point in which $r = 1$, the four base cubics have a common tangent, since, at such a point,

$$f_{ix} : f_{iy} : f_{iz} = f_{jx} : f_{jy} : f_{jz};$$


y = 0
μ = 0
P3
y = 0
ω1
P4
P5
ω'2
ω'1
ω3
P2
P1
z = 0
P6
λ = 0
x = 0
ω2
ω'3

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combining this result with the remark at the end of § 2, we see that, at a fundamental point, the rank r can never be $= 1$. The points, where $r = 1$, are found by equating to zero all determinants of order *two*.

Consider first the determinants

$$(15)\quad f_{1x}f_{3z} - f_{1z}f_{3x} = 0,\quad f_{1x}f_{3y} - f_{1y}f_{3x} = 0,\quad f_{1y}f_{3z} - f_{1z}f_{3y} = 0.$$

Therefore

$$(16)\quad \begin{cases} y^2(\phi_{1x}\phi_{3z} - \phi_{3x}\phi_{1z}) = 0, \\ y\left[\frac{3}{2}y(\phi_{1x}\phi_{3y} - \phi_{3x}\phi_{1y}) + \frac{z}{2}(\phi_{1x}\phi_{3z} - \phi_{3x}\phi_{1z})\right] = 0, \\ y\left[\frac{3}{2}y(\phi_{1y}\phi_{3z} - \phi_{3y}\phi_{1z}) + \frac{x}{2}(\phi_{1x}\phi_{3z} - \phi_{3x}\phi_{1z})\right] = 0, \end{cases}$$

where

$$\phi_{ix} = \frac{\partial \phi_i}{\partial x}, \qquad \phi_{iy} = \frac{\partial \phi_i}{\partial y}, \qquad \phi_{iz} = \frac{\partial \phi_i}{\partial z}.$$

From equations (16), we have either $y = 0$, or simultaneously,

$$(17)\quad \phi_{1x}\phi_{3z} - \phi_{3x}\phi_{1z} = 0,\quad \phi_{1x}\phi_{3y} - \phi_{3x}\phi_{1y} = 0,\quad \phi_{1y}\phi_{3z} - \phi_{3y}\phi_{1z} = 0.$$

The points, which satisfy these equations, are the double points, ω_1, ω_2, ω_3, of the pencil of conics

$$(18)\qquad \alpha\phi_1 + \gamma\phi_3 = 0.$$

The conics, $\phi_1 = 0$, $\phi_3 = 0$, pass through the points, P_2, P_4, P_5, P_6; and the double points, ω_1, ω_2, ω_3, of the pencil, are the three diagonal points of the complete quadrangle $P_2P_4P_5P_6$.

Consider also the determinants

$$(19)\quad f_{2y}f_{4z} - f_{2z}f_{4y} = 0,\ f_{2y}f_{4x} - f_{2x}f_{4y} = 0,\ f_{2x}f_{4z} - f_{2z}f_{4x} = 0.$$

Therefore

$$(20)\quad \begin{cases} x^2(\phi_{2y}\phi_{4z} - \phi_{4y}\phi_{2z}) = 0, \\ x\left[\frac{3}{2}x(\phi_{2y}\phi_{4x} - \phi_{4y}\phi_{2x}) + \frac{z}{2}(\phi_{2y}\phi_{4z} - \phi_{4y}\phi_{2z})\right] = 0, \\ x\left[\frac{3}{2}x(\phi_{2x}\phi_{4z} - \phi_{4x}\phi_{2z}) + \frac{y}{2}(\phi_{2y}\phi_{4z} - \phi_{4y}\phi_{2z})\right] = 0. \end{cases}$$

From these equations, we have either $x = 0$, or simultaneously,

$$(21)\quad \phi_{2y}\phi_{4z} - \phi_{4y}\phi_{2z} = 0,\ \phi_{2y}\phi_{4x} - \phi_{4y}\phi_{2x} = 0,\ \phi_{2x}\phi_{4z} - \phi_{4x}\phi_{2z} = 0;$$

and the points, which satisfy these equations, are the three diagonal points, ω_1', ω_2', ω_3', of the complete quadrangle $P_1P_4P_5P_6$.

It is easily proved:

1) The points, ω_1, ω_2, ω_3, do not lie on the line, $x = 0$.

For, if we suppose one of the points, say ω_1, the intersection of the lines, P_2P_6, P_4P_5, to lie on the line $x = 0$, which passes through the points, P_2, P_3, then the two lines P_2P_3 and P_2P_6 would contain the same point ω_1 (different from P_2) and would coincide; hence, the three points, P_2, P_3, P_6, would lie on the same straight line, against the hypothesis. In like manner, it follows that the points, ω_2, ω_3, do not lie on the line, $x = 0$.

2) Similarly, the points, ω_1', ω_2', ω_3', do not lie on the line, $y = 0$.

3) None of the points, ω_1, ω_2, ω_3, coincides with any of the points, ω_1', ω_2', ω_3'.

If we suppose the point, ω_1, to coincide with any one of the points, say ω_1', then the two lines, P_2P_6, P_5P_6, would contain the same point, ω_1', and would coincide; hence, the three points, P_2, P_5, P_6, would lie on the same straight line. In like manner, it follows that the point, ω_1, does not coincide with either ω_2', or ω_3'. Similarly, we find that neither one of the points, ω_2, ω_3, coincides with any of the points, ω_1', ω_2', ω_3'.

4) The lines, $x = 0$, $y = 0$, meet at the point, P_3; the point, P_3, is therefore the only point in which the six determinants, (15) and (19), vanish. But we have already shown that, at a fundamental point, not all the determinants of order *two* can vanish; therefore, *there are no points in the plane in which* $r = 1$.

$3°$ $\qquad r = 2.$

When $r = 2$, all the determinants of order *three* vanish; whereas, at least, one determinant of order *two* is $\neq 0$.

Such points are the common points of intersection of the Jacobians of the four nets of cubics,

$$(22)\qquad \alpha f_1 + \beta f_2 + \gamma f_3 = 0, \qquad \alpha' f_1 + \beta' f_2 + \delta' f_4 = 0,$$

$$(23)\qquad \alpha''f_1+\gamma''f_3+\delta''f_4=0, \qquad \beta'''f_2+\gamma'''f_3+\delta'''f_4=0.$$

Consider the Jacobian,

$$(24)\qquad J_2=\begin{vmatrix} f_{1x} & f_{1y} & f_{1z}\\ f_{3x} & f_{3y} & f_{3z}\\ f_{4x} & f_{4y} & f_{4z}\end{vmatrix}=\begin{vmatrix} y\phi_{1x} & \phi_1+y\phi_{1y} & y\phi_{1z}\\ y\phi_{3x} & \phi_3+y\phi_{3y} & y\phi_{3z}\\ \phi_4+x\phi_{4x} & x\phi_{4y} & x\phi_{4z}\end{vmatrix},$$

of the net of cubics,

$$(25)\qquad \alpha''f_1+\gamma''f_3+\delta''f_4=0.$$

By an easy reduction, we have

$$(26)\qquad J_2=\tfrac{3}{2}y\left\{xy\begin{vmatrix} \phi_{1x} & \phi_{1y} & \phi_{1z}\\ \phi_{3x} & \phi_{3y} & \phi_{3z}\\ \phi_{4x} & \phi_{4y} & \phi_{4z}\end{vmatrix}+2\phi_4\begin{vmatrix} \phi_1 & \phi_3\\ \phi_{1z} & \phi_{3z}\end{vmatrix}\right\}=0,$$

and finally,

$$(27)\qquad \begin{aligned} J_2=3y\nu\,[&(b_1\lambda-a_1\mu)(c_2\lambda-a_2\nu)(b_2\lambda-a_2\mu)(c_3\mu-b_3\nu)\\ &+a_2\Delta x\mu\nu(b_1\lambda-a_1\mu)+c_2\gamma_3\Delta xy\lambda\mu]=0.\end{aligned}$$

If we interchange μ and ν, b and c, in (11) and (12), f_1, f_2, f_3, f_4 undergo the substitution $(f_1f_3)(f_2f_4)$. Therefore, we obtain for the Jacobian J_4,

$$(28)\qquad J_4=\begin{vmatrix} f_{3x} & f_{3y} & f_{3z}\\ f_{1x} & f_{1y} & f_{1z}\\ f_{2x} & f_{2y} & f_{2z}\end{vmatrix},$$

$$(29)\qquad \begin{aligned} J_4=-3\mu y\,[&(c_1\lambda-a_1\nu)(c_2\lambda-a_2\nu)(b_2\lambda-a_2\mu)(c_3\mu-b_3\nu)\\ &+a_2\Delta x\mu\nu(c_1\lambda-a_1\nu)-b_2\beta_3\Delta xy\lambda\nu]=0.\end{aligned}$$

The Jacobians, J_2, J_4, have the line, $y=0$, as a common factor and intersections at the points, P_1, P_2, P_3, P_4, P_5, P_6, ω_1, ω_2, ω_3, two further points, Q_1, Q_2, and only at these points. For, if we set

$$(30)\qquad \begin{cases} J_2=3y\psi_2=3y\nu\chi_2=0,\\ J_4=-3y\psi_4=-3y\mu\chi_4=0,\end{cases}$$

where

$$(31)\quad \begin{cases} \chi_2 = (b_1\lambda - a_1\mu)(c_2\lambda - a_2\nu)(b_2\lambda - a_2\mu)(c_3\mu - b_3\nu) \\ \qquad\qquad + a_2\Delta x\mu\nu(b_1\lambda - a_1\mu) + c_2\gamma_3\Delta xy\lambda\mu \\ \equiv (\gamma_2 z - \gamma_3 y)(\beta_1 z - \beta_3 x)(\gamma_3 x - \gamma_1 z)(\alpha_1 y - \alpha_2 x) \\ \qquad\qquad + a_2\Delta x\mu\nu(\gamma_2 z - \gamma_3 y) + c_2\gamma_3\Delta xy\lambda\mu, \end{cases}$$

$$(32)\quad \begin{cases} \chi_4 = (c_1\lambda - a_1\nu)(c_2\lambda - a_2\nu)(b_2\lambda - a_2\mu)(c_3\mu - b_3\nu) \\ \qquad\qquad + a_2\Delta x\mu\nu(c_1\lambda - a_1\nu) - b_2\beta_3\Delta xy\lambda\nu \\ \equiv (\beta_3 y - \beta_2 z)(\beta_1 z - \beta_3 x)(\gamma_3 x - \gamma_1 z)(\alpha_1 y - \alpha_2 x) \\ \qquad\qquad + a_2\Delta x\mu\nu(\beta_3 y - \beta_2 z) - b_2\beta_3\Delta xy\lambda\nu, \end{cases}$$

then the points of intersection of $\psi_2 = 0$ and $\psi_4 = 0$ can be determined as follows:

1° $\mu = 0$ and $\nu = 0$ intersect at the point P_4.

2° $\mu = 0$ and $\chi_2 = 0$ intersect at the points, P_4, ω_3, and P_6, P_6, since the point P_6 is a double point on $\chi_2 = 0$.

3° $\nu = 0$ and $\chi_4 = 0$ intersect at the points, P_4, ω_1, and P_5, P_5, since the point P_5 is a double point on $\chi_4 = 0$.

4° $\chi_2 = 0$ and $\chi_4 = 0$ intersect at the points, P_1, P_2, P_3, P_4, P_5, P_6, ω_2, and at the two points, Q_1, Q_2, determined by the equations,

$$(33)\quad \begin{cases} \alpha_2(\beta_1 z - \beta_3 x)(\gamma_3 x - \gamma_1 z) - a_2\Delta(b_1 x + b_3 z)(c_1 x + c_3 z) = 0, \\ y = 0. \end{cases}$$

Each of the points, P_5 and P_6, counts for, at least, two points of intersection, since P_5 is a double point on $\chi_4 = 0$, and P_6 is a double point on $\chi_2 = 0$.

The point P_2 is a double point on both $\chi_2 = 0$ and $\chi_4 = 0$, and counts therefore, at least, for four points of intersection. At P_3, the curves, $\chi_2 = 0$, $\chi_4 = 0$, have coinciding tangents, whose equation is

$$(34)\qquad x(a_2\Delta b_3 c_3 + \alpha_2\beta_1\gamma_1) - y\alpha_1\beta_1\gamma_1 = 0.$$

At P_1, the equations of the tangents to $\chi_4 = 0$, $\chi_2 = 0$ are,

respectively,

$$(35)\quad \begin{cases} \beta_3 y(1-\rho b_2 c_1) - \beta_2 z = 0, \text{ where } \rho \equiv \dfrac{\Delta a_1}{\beta_3\gamma_3\alpha_2 + a_2 b_1 c_1 \Delta}, \\ \gamma_3 y(1-\rho b_1 c_2) - \gamma_2 z = 0, \end{cases}$$

and because

$$(36)\quad \begin{vmatrix} \beta_3(1-\rho b_2 c_1) & \beta_2 \\ \gamma_3(1-\rho b_1 c_2) & \gamma_2 \end{vmatrix} = 0,$$

the two tangents coincide. Hence each of the points, P_3 and P_1, counts, at least, for two points of intersection.

Counting each point of intersection with the minimum multiplicity thus determined, we obtain a total of *sixteen* points of intersection, *the exact number of points of intersection of the two biquadratics*, $\chi_2 = 0$, $\chi_4 = 0$. Hence, the two biquadratics, $\chi_2 = 0$, $\chi_4 = 0$, can have no other points of intersection than those enumerated above; and *the Jacobians*, J_2, J_4, *have the line*, $y = 0$, *as a common factor, and intersections at the points*, P_1, P_2, P_3, P_4, P_5, P_6, ω_1, ω_2, ω_3, Q_1, Q_2, *and only at these points.*

If we interchange x and y, subscripts 1 and 2, in (11) and (12), f_1, f_2, f_3, f_4 undergo the substitution $(f_1\, f_2)(f_3\, f_4)$. Therefore we obtain, for the Jacobians, J_1, J_3,

$$(37)\quad \begin{aligned} J_1 = 3x\nu[&(b_2\lambda - a_2\mu)(b_1\lambda - a_1\mu)(c_1\lambda - a_1\nu)(c_3\mu - b_3\nu) \\ &- a_1\Delta y\mu\nu(b_2\lambda - a_2\mu) + c_1\gamma_3\Delta xy\lambda\mu] = 0, \end{aligned}$$

$$(38)\quad \begin{aligned} J_3 = -3x\mu[&(c_2\lambda - a_2\nu)(b_1\lambda - a_1\mu)(c_1\lambda - a_1\nu)(c_3\mu - b_3\nu) \\ &- a_1\Delta y\mu\nu(c_2\lambda - a_2\nu) - b_1\beta_3\Delta xy\lambda\nu] = 0. \end{aligned}$$

The Jacobians, J_3, J_1, have the line, $x = 0$, as a common factor, and intersections at the points, P_1, P_2, P_3, P_4, P_5, P_6, ω_1', ω_2', ω_3', Q_1', Q_2', and only at these points; Q_1', Q_2', are on $x = 0$; and since none of the points, ω_1', ω_2', ω_3', Q_1', Q_2', coincides with any one of the points, ω_1, ω_2, ω_3, Q_1, Q_2, we have the lemma:

There are six, and only six, points in the plane Π *in which* $r = 2$; *namely, the six fundamental points,* P_i, *and* $r = 3$, AT ALL OTHER POINTS.

CHAPTER II.

The One-to-One Correspondence Between the Cubic Surface and the Plane.

§ 4. Transformation from plane to cubic surface.

As in Chapter 1, let $P_1, \cdots, P_6$ be six distinct points in a plane Π, no three of which lie on a straight line, and further let

$$(1) \qquad \begin{aligned} f_1(\lambda, \mu, \nu) &= 0, \qquad f_2(\lambda, \mu, \nu) = 0, \\ f_3(\lambda, \mu, \nu) &= 0, \qquad f_4(\lambda, \mu, \nu) = 0, \end{aligned}$$

where λ, μ, ν denote any system of trilinear coördinates, be the four base cubics of the triply infinite system passing through the six points.

Then, if $x_1 : x_2 : x_3 : x_4$ denote tetrahedral coördinates of a point in space, the equations,

$$(2) \qquad \rho x_1 = f_1, \qquad \rho x_2 = f_2, \qquad \rho x_3 = f_3, \qquad \rho x_4 = f_4,$$

define, in parameter representation, a cubic surface F_3 and, at the same time, a correspondence * between the points of the plane and the points of the cubic surface.

a) To any point, P_0, in the plane Π, different from the points, $P_1, \cdots, P_6$, corresponds one point, P_0', on the surface F_3; since, not all four of the quantities, $f_1^0, f_2^0, f_3^0, f_4^0$, equal zero, and the ratios,

$$x_1 : x_2 : x_3 : x_4 = f_1^0 : f_2^0 : f_3^0 : f_4^0,$$

have definite values; $f_1^0, f_2^0, f_3^0, f_4^0$ are the values of the functions, f_1, f_2, f_3, f_4, respectively, at the point P_0. The six points, $P_1, \cdots, P_6$, for which the functions, f_1, f_2, f_3, f_4, vanish simultaneously, are called *fundamental points.*

b) To a fundamental point, together with the totality of the paths of approach to it, corresponds a fundamental curve on F_3.

* Clebsch, "Geometrie auf den Flächen dritter Ordnung," *Crelle*, 65.

Let

$$\lambda = \lambda_0 + \lambda_1 t, \qquad \mu = \mu_0 + \mu_1 t, \qquad \nu = \nu_0 + \nu_1 t, \tag{3}$$

$$\left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0,^* \tag{4}$$

be a path of approach, L, to a fundamental point, P_0. At P_0,

$$f_1^0 = f_2^0 = f_3^0 = f_4^0 = 0,$$

but according to § 3, not all of the quantities, $f_{1\lambda}^0, \cdots, f_{4\nu}^0, = 0$;

$$f_{1\lambda}^0 = \left(\frac{\partial f_1}{\partial \lambda}\right)_{\lambda=\lambda_0}, \cdots, f_{4\nu}^0 = \left(\frac{\partial f_4}{\partial \nu}\right)_{\nu=\nu_0}.$$

Expanding the functions, f_1, f_2, f_3, f_4, in the vicinity of P_0, we get

$$\rho x_j = f_{j\lambda}^0(\lambda-\lambda_0) + f_{j\mu}^0(\mu-\mu_0) + f_{j\nu}^0(\nu-\nu_0) + \cdots, \ (j=1, 2, 3, 4), \tag{5}$$

and by (3) $\therefore$

$$\sigma x_j = \lambda_1 f_{j\lambda}^0 + \mu_1 f_{j\mu}^0 + \nu_1 f_{j\nu}^0 + \text{powers of } t, \qquad \sigma = \frac{\rho}{t}. \tag{6}$$

If we approach to the limit, $t = 0$, the ratios, $x_1 : x_2 : x_3 : x_4$, approach to definite values,

$$\sigma x_j = \lambda_1 f_{j\lambda}^0 + \mu_1 f_{\mu j}^0 + \nu_1 f_{j\nu}^0, \tag{7}$$

and define a definite point. For, suppose

$$\lambda_1 f_{j\lambda}^0 + \mu_1 f_{j\mu}^0 + \nu_1 f_{j\nu}^0 = 0, \qquad \text{for} \qquad j = 1, 2, 3, 4. \tag{7a}$$

Then, since by § 3 at a fundamental point the rank of the matrix of the coefficients is *two*, these equations define a unique solution for $\lambda_1 : \mu_1 : \nu_1$; and, since

$$\lambda_0 f_{j\lambda}^0 + \mu_0 f_{j\mu}^0 + \nu_0 f_{j\nu}^0 = 3 f_j^0 = 0,$$

it follows, that $\lambda_0 : \mu_0 : \nu_0$ is a solution for the same equations; hence,

$$\lambda_1 : \mu_1 : \nu_1 = \lambda_0 : \mu_0 : \nu_0,$$

against the assumption (4).

* By this notation, we mean that, at least, one of the determinants, $\lambda_0\mu_1 - \mu_0\lambda_1$, $\lambda_0\nu_1 - \nu_0\lambda_1$, $\mu_0\nu_1 - \nu_0\mu_1$, is different from zero.

c) Two different lines of approach to P_0 determine two different points on the fundamental curve.

Let

$$\lambda = \lambda_0 + \lambda_1' t,\ \mu = \mu_0 + \mu_1' t,\ \nu = \nu_0 + \nu_1' t, \tag{8}$$

$$\left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1' & \mu_1' & \nu_1' \end{matrix} \right\| \neq 0,$$

be another path of approach, L', to P_0, not coinciding with L, that is

$$\begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_1' & \mu_1' & \nu_1' \end{vmatrix} \neq 0. \tag{9}$$

If the two points, P_0', P_0'', on the fundamental curve corresponding to the lines L, L', coincide, then

$$\rho(\lambda_1 f_{j\lambda}^0 + \mu_1 f_{j\mu}^0 + \nu_1 f_{j\nu}^0) = \lambda_1' f_{j\lambda}^0 + \mu_1' f_{j\mu}^0 + \nu_1' f_{j\nu}^0 \quad (j=1, 2, 3, 4). \tag{10}$$

Therefore,

$$(\rho\lambda_1 - \lambda_1') f_{j\lambda}^0 + (\rho\mu_1 - \mu_1') f_{j\mu}^0 + (\rho\nu_1 - \nu_1') f_{j\nu}^0 = 0. \tag{11}$$

By § 3, the rank of the matrix of the coefficients in (11) is $r = 2$, and, by the Euler equations, we know $\lambda_0 : \mu_0 : \nu_0$ is a solution of these equations; hence, it follows, that

$$\begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_1' & \mu_1' & \nu_1' \end{vmatrix} = 0, \tag{12}$$

against the assumption (9).

The point, P_0', describes the fundamental curve as the parameters, λ_1, μ_1, ν_1, take all possible values.

d) A fundamental curve on F_3 is a straight line.

For, according to § 3, the rank of the matrix,

$$\| f_{j\lambda}^0 \quad f_{j\mu}^0 \quad f_{j\nu}^0 \| \qquad (j=1, 2, 3, 4), \tag{13}$$

is $r=2$. Hence, if we suppose, for instance, that $f^0_{1\mu}f^0_{2\nu}-f^0_{2\mu}f^0_{1\nu}\neq 0$, two factors m' and n' can be found, so that,

$$f^0_{j\lambda}=m'f^0_{j\mu}+n'f^0_{j\nu};$$

and these values substituted in (7) give

$$\sigma x_j=f^0_{j\mu}(m'\lambda_1+\mu_1)+f^0_{j\nu}(n'\lambda_1+\nu_1). \tag{14}$$

If we set $m'\lambda+\mu_1=t_1$, $n'\lambda_1+\nu_1=t_2$, we have

$$\sigma x_j=f^0_{j\mu}t_1+f^0_{j\nu}t_2, \tag{15}$$

the equations of a straight line in parameter representation. There are, therefore, six fundamental lines on F_3 corresponding to the six fundamental points, $P_1,\cdots,P_6$, in the plane, Π.

§ 5. The one-to-one character of the correspondence between the plane and the cubic surface.

a) To two non-fundamental points, in the plane Π, *always correspond two different points on* F_3, *provided the additional condition be imposed upon the fundamental points, that the six points,* $P_1,\cdots,P_6$, *do not lie on a conic.*

If to a non-fundamental point, P_0, corresponds the same point on F_3 as to the non-fundamental point, P, then

$$\left\|\begin{matrix} f^0_1 & f^0_2 & f^0_3 & f^0_4 \\ f_1 & f_2 & f_3 & f_4 \end{matrix}\right\|=0. \tag{16}$$

The quantities, $f^0_1, f^0_2, f^0_3, f^0_4$, are not all equal to zero; suppose $f^0_1\neq 0$; (16) is then equivalent * to the three equations,

$$f^0_1f_2-f^0_2f_1=0,\ f^0_1f_3-f^0_3f_1=0,\ f^0_1f_4-f^0_4f_1=0. \tag{17}$$

These cubics have either a common factor or no common factor.

1° A common factor must be of degree, $n=1, 2, 3$.

1) $$n=1.$$

Let the greatest common factor L of the three cubics be of degree

* Kronecker, "Bemerkungen zur Determinanten-Theorie," *Crelle* 72, p. 152.

$n = 1$, in λ, μ, ν; then, by (17), we have

$$(18) \qquad \begin{cases} f_1^0 f_2 - f_2^0 f_1 \equiv L \cdot \chi_1', \\ f_1^0 f_3 - f_3^0 f_1 \equiv L \cdot \chi_2', \\ f_1^0 f_4 - f_4^0 f_1 \equiv L \cdot \chi_3', \end{cases}$$

where $\chi_1', \chi_2', \chi_3'$ are quadratic factors.

Since no three fundamental points, $P_1, \cdots, P_6$, lie on a straight line, then, if two of them lie on $L = 0$, must the remaining four lie on each conic, $\chi_1' = 0$, $\chi_2' = 0$, $\chi_3' = 0$; because each cubic in (18) passes through each fundamental point. Therefore $\chi_1', \chi_2', \chi_3'$ belong to the same pencil; but this would involve a linear relation between f_1, f_2, f_3, f_4, which is against our hypothesis.

If only one fundamental point should lie on $L = 0$, then the remaining five fundamental points must all lie on each conic $\chi_1', \chi_2', \chi_3'$; hence $\chi_1', \chi_2', \chi_3'$ would coincide, and therefore the common factor in (18) would be of higher degree than $n = 1$, against the assumption.

If no fundamental point should lie on the line $L = 0$, then all six fundamental points must lie on each conic, $\chi_1' = 0$, $\chi_2' = 0$, $\chi_3' = 0$, against the hypothesis.

2) $\qquad n = 2.$

Let the greatest common factor ψ of the three cubics be of degree $n = 2$, in λ, μ, ν; then, from (17), we have

$$(19) \qquad \begin{cases} f_1^0 f_2 - f_2^0 f_1 \equiv \psi \cdot l_1, \\ f_1^0 f_3 - f_3^0 f_1 \equiv \psi \cdot l_2, \\ f_1^0 f_4 - f_4^0 f_1 \equiv \psi \cdot l_3, \end{cases}$$

where l_1, l_2, l_3 are linear factors. Since all six points, $P_1, \cdots, P_6$, do not lie on a conic, then, if five of them should lie on $\psi = 0$, the remaining one must lie on each line, $l_1 = 0$, $l_2 = 0$, $l_3 = 0$; hence a linear relation must exist between l_1, l_2, l_3, and therefore between f_1, f_2, f_3, f_4, against the hypothesis. If less than five fundamental points should lie on $\psi = 0$, then the remaining ones must lie on each line, $l_1 = 0$, $l_2 = 0$, $l_3 = 0$; hence these lines would coincide,

and therefore the common factor, in (19), would be of degree $n = 3$, against the assumption.

In like manner, it is easily proved, if $n = 3$, that a linear relation exists between f_1, f_2, f_3, f_4, in which not all the coefficients equal zero. Therefore, it follows, that the three cubics (17) do not contain any common factor.

2° Since the three cubics, in (17), do not contain any common factor, they can intersect in only a finite number of points; we know already seven points of intersection, $P_0, P_1, \cdots, P_6$.

Suppose one additional point P, different from the seven points, $P_0, P_1, \cdots, P_6$, to be a point of intersection of these cubics, then they meet in eight points and therefore have a ninth base point in common; hence, they belong to the same pencil and one of them is expressible linearly in terms of the other two.

Set

$$(20)\quad \rho_2(f_1^0 f_2 - f_2^0 f_1) + \rho_3(f_1^0 f_3 - f_3^0 f_1) + \rho_4(f_1^0 f_4 - f_4^0 f_1) = 0,$$

where ρ_2, ρ_3, ρ_4 are not all equal to zero; therefore

$$(21)\quad -f_1(\rho_2 f_2^0 + \rho_3 f_3^0 + \rho_4 f_4^0) + \rho_2 f_1^0 f_2 + \rho_3 f_1^0 f_3 + \rho_4 f_1^0 f_4 = 0,$$

since ρ_2, ρ_3, ρ_4 are not all equal to zero and $f_1^0 \neq 0$, the coefficients of this linear relation can not all be zero; but this is a contradiction to the previous result that f_1, f_2, f_3, f_4 are linearly independent.

b) The image of a non-fundamental point does not lie on a fundamental line.

For, if the image of the non-fundamental point, P, coincides with the point on the fundamental line corresponding to a fundamental point, $P_0(\lambda_0, \mu_0, \nu_0)$, determined by the line* of approach,

$$(22)\quad \lambda = \lambda_0 + \lambda_1 t, \quad \mu = \mu_0 + \mu_1 t, \quad \nu = \nu_0 + \nu_1 t,$$

$$\begin{Vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{Vmatrix} \neq 0,$$

then

$$(23)\quad \begin{Vmatrix} f_1 & f_2 & f_3 & f_4 \\ S_1 & S_2 & S_3 & S_4 \end{Vmatrix} = 0,$$

*Compare § 4, *b*.

where

$$S_j = \lambda_1 f^0_{j\lambda} + \mu_1 f^0_{j\mu} + \nu_1 f^0_{j\nu} \qquad (j=1,2,3,4); \tag{24}$$

not all of the quantities S_1, S_2, S_3, S_4 are equal to zero.

Suppose $S_1 \neq 0$; then (23) * is equivalent to the three equations,

$$f_1 S_2 - f_2 S_1 = 0,\ f_1 S_3 - f_3 S_1 = 0,\ f_1 S_4 - f_4 S_1 = 0. \tag{25}$$

It follows then exactly as under *a*) that these three cubics can have no common factor; they intersect therefore in a finite number of points, and we know already seven points of intersection; namely, the six fundamental points, one of which, P_0, counts twice, since, at the point P_0, the line (22) meets each cubic in two coinciding points; for, if we substitute λ, μ, ν of (22) in (25) and expand according to powers of t, the expansions begin with the second power of t. Hence we infer as under *a*) that the three cubics (25) can have no further point in common.

c) *The image of a non-fundamental point is not a singular point on the surface* F_3.

Let P_0 be any non-fundamental point and consider the line L through P_0,

$$\lambda = \lambda_0 + \lambda_1 t,\ \mu = \mu_0 + \mu_1 t,\ \nu = \nu_0 + \nu_1 t, \tag{26}$$

$$\left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0. \tag{27}$$

The image P'_0 of P_0 is given by

$$x_1 : x_2 : x_3 : x_4 = f^0_1 : f^0_2 : f^0_3 : f^0_4, \tag{28}$$

and the tangent to the image of L at P'_0 by

$$x_1 = f^0_1 + S_1 t,\ x_2 = f^0_2 + S_2 t,\ x_3 = f^0_3 + S_3 t,\ x_4 = f^0_4 + S_4 t. \tag{29}$$

By the Euler equations, we get

$$3 f^0_j = \lambda_0 f^0_{j\lambda} + \mu_0 f^0_{j\mu} + \nu_0 f^0_{j\nu} \qquad (j=1,2,3,4) \tag{30}$$

and substituting (24) and (30) in (29), we get

$$\rho'' x_j = f^0_{j\lambda}(\lambda_0 + 3\lambda_1 t) + f^0_{j\mu}(\mu_0 + 3\mu_1 t) + f^0_{j\nu}(\nu_0 + 3\nu_1 t). \tag{31}$$

* Kronecker, *Crelle*, 72.

Set $\lambda_0 + 3\lambda_1 t = \lambda'$, $\mu_0 + 3\mu_1 t = \mu'$, $\nu_0 + 3\nu_1 t = \nu'$, therefore

$$\rho'' x_j = \lambda' f^0_{j\lambda} + \mu' f^0_{j\mu} + \nu' f^0_{j\nu}. \tag{32}$$

Since, according to § 3, the rank of the matrix, $\| f^0_{j\lambda}\, f^0_{j\mu}\, f^0_{j\nu} \|$, is $r = 3$, the equations (32) represent, in parameter representation with the parameters λ', μ', ν', a fixed plane through P'_0, independent of λ_1, μ_1, ν_1, and the tangent (29) lies in this plane. The surface F_3 has therefore at P'_0 a determinate tangent plane and the point P'_0 is a non-singular point.

d) Two fundamental lines on the surface F_3 have no common point.

If the fundamental line L'_1 corresponding to P_1 and the fundamental line L'_3 corresponding to P_3 have a common point P' determined on L'_1 by the path of approach to $P_1(1, 0, 0)$, (where λ, μ, ν are trilinear coördinates with respect to the triangle of reference $P_1P_2P_3$, these three fundamental points are its vertices)

$$\lambda = 1+\lambda'_1 t, \quad \mu = 0+\mu'_1 t, \quad \nu = 0+\nu'_1 t, \tag{33}$$

$$\left\| \begin{matrix} 1 & 0 & 0 \\ \lambda'_1 & \mu'_1 & \nu'_1 \end{matrix} \right\| \neq 0,$$

and on L'_3 by the path of approach to $P_3(0, 0, 1)$,

$$\lambda = 0+\lambda_1 t, \quad \mu = 0+\mu_1 t, \quad \nu = 1+\nu_1 t, \tag{34}$$

$$\left\| \begin{matrix} 0 & 0 & 1 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0,$$

then

$$\left\| \begin{matrix} S^{(3)}_1 & S^{(3)}_2 & S^{(3)}_3 & S^{(3)}_4 \\ S^{(1)}_1 & S^{(1)}_2 & S^{(1)}_3 & S^{(1)}_4 \end{matrix} \right\| = 0, \tag{35}$$

where

$$\begin{cases} S^{(3)}_j = \lambda_1 f^{(3)}_{j\lambda} + \mu_1 f^{(3)}_{j\mu} + \nu_1 f^{(3)}_{j\nu}, \\ f^{(3)}_{j\lambda} = \left(\dfrac{\partial f_j}{\partial \lambda}\right)_{\lambda=0}, \cdots, f^{(3)}_{j\nu} = \left(\dfrac{\partial f_j}{\partial \nu}\right)_{\nu=1}; \\ S^{(1)}_j = \lambda'_1 f^{(1)}_{j\lambda} + \mu'_1 f^{(1)}_{j\mu} + \nu'_1 f^{(1)}_{j\nu}, \\ f^{(1)}_{j\lambda} = \left(\dfrac{\partial f_j}{\partial \lambda}\right)_{\lambda=1}, \cdots, f^{(1)}_{j\nu} = \left(\dfrac{\partial f_j}{\partial \nu}\right)_{\nu=0} \quad (j=1, 2, 3, 4). \end{cases} \tag{35$_a$}$$

By the canonical form of f_1, f_2, f_3, f_4, we find,

$$(35_b)\quad \begin{cases} S_1^{(3)} = \mu_1 b_3 \beta_1, & S_2^{(3)} = -\lambda_1 b_3 \beta_2, \\ S_3^{(3)} = -\mu_1 c_3 \gamma_1, & S_4^{(3)} = \lambda_1 c_3 \gamma_2, \\ S_1^{(1)} = -\mu_1' b_1 \beta_3, & S_2^{(1)} = \mu_1' b_1 \beta_3 - \nu_1' b_1 \beta_2, \\ S_3^{(1)} = \mu_1' c_1 \gamma_3, & S_4^{(1)} = -\mu_1' c_1 \gamma_3 + \nu_1' c_1 \gamma_2. \end{cases}$$

Since not both λ_1, μ_1 equal to zero, suppose $\mu_1 \neq 0$, therefore $S_1^{(3)} \neq 0$; then (35) is equivalent to the three equations,

$$(36)\quad S_1^{(3)} S_2^{(1)} - S_2^{(3)} S_1^{(1)} = 0, \quad S_1^{(3)} S_3^{(1)} - S_3^{(3)} S_1^{(1)} = 0, \quad S_1^{(3)} S_4^{(1)} - S_4^{(3)} S_1^{(1)} = 0.$$

Therefore,

$$(37)\quad \begin{cases} (\mu_1 \beta_1 \beta_3 - \lambda_1 \beta_2 \beta_3) \mu_1' - \mu_1 \beta_1 \beta_2 \nu_1' = 0, & \text{I}; \\ (b_3 c_1 \beta_1 \gamma_3 - b_1 c_3 \beta_3 \gamma_1) \mu_1' = 0, & \text{II}; \\ (\lambda_1 b_1 c_3 \beta_3 \gamma_2 - \mu_1 b_3 c_1 \beta_1 \gamma_3) \mu_1' + \mu_1 b_3 c_1 \beta_1 \gamma_2 \nu_1' = 0, & \text{III}. \end{cases}$$

In I and III, if $\mu_1' = 0$, then $\nu_1' = 0$, since $\mu_1 \neq 0$; against the assumption in (33); therefore, we have $\mu_1' \neq 0$.

In II, since $\mu_1' \neq 0$, we have

$$(38)\quad b_3 c_1 \beta_1 \gamma_3 - b_1 c_3 \beta_3 \gamma_1 \equiv \begin{vmatrix} a_1 b_1 & a_1 c_1 & b_1 c_1 \\ a_2 b_2 & a_2 c_2 & b_2 c_2 \\ a_3 b_3 & a_3 c_3 & b_3 c_3 \end{vmatrix} = 0,$$

the necessary and sufficient condition for all six fundamental points to lie on a conic, against the hypothesis. In like manner, it follows, if $\mu_1 = 0$, that all six fundamental points lie on a conic. What has just been proved for the two fundamental points P_1, P_3 holds for any two fundamental points. Hence *two fundamental lines on the surface F_3 have no common point.*

As a result of these theorems, it follows, *that to every point of F_3, not on a fundamental line, corresponds one, and but one, point in* Π, WITHOUT EXCEPTION; *and to every point of F_3, on a fundamental line, corresponds, in* Π, *one, and but one, fundamental point together with one, and but one, path of approach to it,* WITHOUT EXCEPTION;

there are, therefore, no fundamental points on F_3 and no fundamental curves in the plane II.

§ 6. DETERMINATION OF ALL STRAIGHT LINES ON F_3.

Let

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0, \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = 0, \end{cases} \tag{39}$$

where

$$\left\| \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix} \right\| \neq 0, \tag{40}$$

be any straight line, L', in space different from one of the fundamental lines; the parameters λ, μ, ν of its points of intersection with the surface F_3, are the coördinates of the points of intersection, different from the fundamental points, of the two cubics

$$\begin{cases} \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 = 0, \\ \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4 = 0, \end{cases} \tag{41}$$

in the plane II.

If the line, L', is to lie on the surface F_3, these two cubics must have an infinity of points in common; they must therefore have a common factor of degree, $n = 3, 2, 1$.

1) Let ϕ be a common factor of degree, $n = 3$, in λ, μ, ν. Set

$$\begin{cases} \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 = \bar{\rho}_1 \phi, \ \bar{\rho}_1 \neq 0, \\ \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4 = \bar{\rho}_2 \phi, \ \rho_2 \neq 0; \end{cases} \tag{42}$$

therefore

$$\begin{aligned} (\bar{\rho}_2 \alpha_1 - \bar{\rho}_1 \beta_1) f_1 + (\bar{\rho}_2 \alpha_2 - \rho_1 \beta_2) f_2 + (\bar{\rho}_2 \alpha_3 - \bar{\rho}_1 \beta_3) f_3 \\ + (\bar{\rho}_2 \alpha_4 - \bar{\rho}_1 \beta_4) f_4 = 0, \end{aligned} \tag{43}$$

which is a linear relation, in which on account of (40) not all the coefficients vanish, against the hypothesis.

2) Suppose the greatest common divisor ψ of the two cubics to be of degree, $n = 2$, in λ, μ, ν.

Set

$$\begin{cases} \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 = \psi \cdot L_1, \\ \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4 = \psi \cdot L_2, \end{cases} \tag{44}$$

where L_1, L_2 are linear factors.

Now the two cubics pass through the six fundamental points, which do not all lie on a conic. Hence the conic, $\psi = 0$, can pass through at most five of the six fundamental points. On the other hand, not less than five of the fundamental points can lie on $\psi = 0$; for, if there were four, the lines L_1, L_2 would pass through the two remaining points and would therefore coincide, and the two cubics would have a common factor of degree, $n = 3$; if there were less than four, three or more fundamental points would lie on a straight line.

Hence ψ must pass through exactly five of the fundamental points.

Conversely, if $\psi = 0$ be any conic passing through five of the six fundamental points, and L_1, L_2 are two distinct straight lines intersecting at the sixth fundamental point, then the two cubics, $\psi \cdot L_1 = 0$, $\psi \cdot L_2 = 0$, belong to the linear system of cubics through the points, $P_1, \cdots, P_6$, and are therefore expressible in the form (44).

To the points of the conic $\psi = 0$ corresponds then a straight line on the surface F_3. Now six, and only six, different conics can be described to pass through five, at a time, of the six points, $P_1, \cdots, P_6$; hence the conic $\psi = 0$ can occupy six different positions and correspondingly, we obtain six straight lines on the surface F_3.

3) Let the greatest common factor of the two cubics be of degree, $n = 1$, in λ, μ, ν.

Set

$$(45) \qquad \begin{cases} \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 = L \cdot \chi_1, \\ \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4 = L \cdot \chi_2, \end{cases}$$

where χ_1, χ_2 are quadratic factors.

Since no three fundamental points lie on a line, at least, four of them are external to $L = 0$, these four must lie on χ_1, χ_2 and the remaining two must lie on $L = 0$. Conversely, every straight line through two of the six fundamental points gives rise to a straight line on the surface. We thus obtain fifteen straight lines on the surface F_3, which together with the six fundamental lines and the six lines found under *b*) give a total of twenty-seven. Therefore, *there are twenty-seven, and only twenty-seven, straight lines on the surface* F_3.

CHAPTER III.

The Transformation of an Algebraic Curve from the Plane Π Upon the Cubic Surface.

§ 7. Generalities about cycles.

Let

$$K: \quad G(\lambda, \mu, \nu) = 0, \tag{1}$$

be an algebraic curve in the plane Π, then, according to the theory* of the singularities of algebraic curves, all points of K in the vicinity of a given point $P_0(\lambda_0, \mu_0, \nu_0)$ of K are expressible by means of a finite number of convergent power series of an auxiliary variable t (the parameter) as follows:

$$\left\{\begin{aligned} \rho\lambda &= \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots, \\ \rho\mu &= \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots, \\ \rho\nu &= \nu_0 + \nu_1 t + \nu_2 t^2 + \cdots, \end{aligned}\right. \tag{2}$$

(C)

called, according to C. Jordan† "cycles." Jordan further shows (Cours D'Analyse t. I, p. 563) that the parameters of the points of intersection of a straight line

$$A\lambda + B\mu + C\nu = 0, \tag{3}$$

with the cycle (C) are given by the equation

$$A\lambda_0 + B\mu_0 + C\nu_0 + (A\lambda_1 + B\mu_1 + C\nu_1)t + \cdots = 0. \tag{4}$$

If the line passes through the point P_0, the "center of the cycle," we have

$$A\lambda_0 + B\mu_0 + C\nu_0 = 0, \tag{5}$$

* Puiseux, "Recherches sur les fonctions algébriques," Liouville, *Journal de Mathematiques*, t. 15, p. 365; Weierstrass, "Theorie der Abelschen Transcendenten, Bd. 4, p. 13; Hamburger, "Ueber die Entwickelung algebraischer Functionen in Reihen," *Zeitschrift für Mathematik und Physik*, Bd. 16, p. 461.

† C. Jordan, "Cours D'Analyse," t. I, p. 561.

and if λ_r, μ_r, ν_r is the first column of coefficients in the expansion, which are not proportional to λ_0, μ_0, ν_0, the expansion (4) begins with the term $(A\lambda_r + B\mu_r + C\nu_r)t^r$. Therefore,

$$(A\lambda_r + B\mu_r + C\nu_r)t^r + \cdots = 0, \tag{6}$$

and the line (3) has, at P_0, r coinciding points in common with the cycle (C); when the coefficients A, B, C satisfy besides (5) the equation

$$A\lambda_r + B\mu_r + C\nu_r = 0, \tag{7}$$

the line, whose equation is

$$\begin{vmatrix} \lambda & \mu & \nu \\ \lambda_0 & \mu_0 & \nu_0 \\ \lambda_r & \mu_r & \nu_r \end{vmatrix} = 0, \tag{8}$$

has, at P_0, more than r coinciding points in common with the cycle. The number r is called the *order* of the cycle and the line (8) is the *tangent* to the cycle.

If the point P_0 is an ordinary point of K, then all the points of K in the vicinity of P_0 are expressible by *one* single cycle of order *one* (a linear cycle). If P_0 is an "ordinary" multiple point of order σ, all the points of K in the vicinity of P_0 are expressible by means of σ linear cycles with distinct tangents.

These definitions concerning cycles can easily be extended to curves in space; let

$$\begin{cases} \rho x_1 = K_0 + K_1 t + K_2 t^2 + \cdots, \\ \rho x_2 = L_0 + L_1 t + L_2 t^2 + \cdots, \\ \rho x_3 = M_0 + M_1 t + M_2 t^2 + \cdots, \\ \rho x_4 = N_0 + N_1 t + N_2 t^2 + \cdots, \end{cases} \tag{9}$$

(C')

be a cycle in space, where K_0, L_0, M_0, N_0, the coördinates of P_0', the center of the cycle, are not all equal to zero. Then the parameters of the points of intersection of an arbitrary plane

$$Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0, \tag{10}$$

with the cycle (C') are given by the equation

$$(11)\quad AK_0+BL_0+CM_0+DN_0+(AK_1+BL_1+CM_1+DN_1)t+\cdots=0.$$

If the plane passes through the center P_0' of the cycle (C'), then

$$(12)\qquad AK_0+BL_0+CM_0+DN_0=0,$$

and if K_r, L_r, M_r, N_r is the first column of coefficients in the expansion (11), which are not proportional to K_0, L_0, M_0, N_0, the expansion begins with the term $(AK_r+BL_r+CM_r+DN_r)t^r$. Therefore,

$$(13)\qquad (AK_r+BL_r+CM_r+DN_r)t^r+\cdots=0,$$

and the plane (10) has, at P_0', r coinciding points in common with the cycle (C'); when the coefficients A, B, C, D satisfy besides (12) the equation

$$(14)\qquad AK_r+BL_r+CM_r+DN_r=0,$$

where

$$(15)\qquad \left\|\begin{matrix} K_0 & L_0 & M_0 & N_0 \\ K_r & L_r & M_r & N_r \end{matrix}\right\| \neq 0,$$

then we can solve equations (12) and (14) for two of the quantities, say A and B, in terms of the others, C and D; hence, by substituting the values thus obtained in (10), we get a pencil of planes, whose common axis is given by the equations

$$(16)\qquad \begin{vmatrix} x_1 & K_0 & K_r \\ x_2 & L_0 & L_r \\ x_3 & M_0 & M_r \end{vmatrix}=0, \qquad \begin{vmatrix} x_1 & K_0 & K_r \\ x_2 & L_0 & L_r \\ x_4 & N_0 & N_r \end{vmatrix}=0;$$

and, in which case, the line (16) has, at P_0', more than r coinciding points in common with the cycle (C'). The number r is called the *order* of the cycle, and the line (16), in parameter representation

$$(17)\qquad \begin{aligned} \rho x_1 &= K_0+K_r u, & \rho x_2 &= L_0+L_r u, \\ \rho x_3 &= M_0+M_r u, & \rho x_4 &= N_0+N_r u, \end{aligned}$$

is the *tangent* to the cycle.

§ 8. Transformation of the non-singular points of the algebraic curve K to the cubic surface.

We now apply the results of the preceding sections to the resolution of the singular points of an algebraic curve into double points.

Let

$$K: \quad G(\lambda, \mu, \nu) = 0, \tag{18}$$

be the algebraic curve in the plane Π; whose singularities we wish to resolve. We suppose that the curve K is *irreducible* and that it has r *ordinary multiple points*, $A_1, \cdots, A_r$, with only linear cycles and with distinct tangents, and no other singular points.* We may further suppose without loss of generality that the degree of the curve K is greater than three.

We let one of the six fundamental points coincide with the multiple point, A_1, to be resolved, but choose all other fundamental points external to K. The image of K is a curve K' on F_3; we propose to study the properties of this curve K'.

Let P_0' be any point of the curve K'; according to § 5, P_0' is the image of one, and but one, point P_0 of K. The point P_0 is either an ordinary point, or one of the multiple points, $A_2, \cdots, A_r$, or the multiple point A_1. If P_0 is an ordinary point, then all points of K in the vicinity of P_0 are represented by one linear cycle (C),

$$\left\{\begin{aligned} \rho\lambda &= \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots, \\ \rho\mu &= \mu_0 + \mu_1 t + \mu_2 t^2 + \cdots, \\ \rho\nu &= \nu_0 + \nu_1 t + \nu_2 t^2 + \cdots, \end{aligned}\right. \qquad (C) \tag{19}$$

where not all of the quantities λ_0, μ_0, ν_0 equal zero and

$$\left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0. \tag{20}$$

The image of the cycle (C) is a cycle (C'),

$$\left\{\begin{aligned} \rho' x_1 &= K_0 + K_1 t + K_2 t^2 + \cdots, \\ \rho' x_2 &= L_0 + L_1 t + L_2 t^2 + \cdots, \\ \rho' x_3 &= M_0 + M_1 t + M_2 t^2 + \cdots, \\ \rho' x_4 &= N_0 + N_1 t + N_2 t^2 + \cdots, \end{aligned}\right. \qquad (C') \tag{21}$$

* Noether, *Mathematische Annalen*, Bd. 9, p. 166.

a cycle of the curve K'; where

$$(22)\quad \begin{cases} K_0 = f_1^0,\ K_1 = \lambda_1 f_{1\lambda}^0 + \mu_1 f_{1\mu}^0 + \nu_1 f_{1\nu}^0, \cdots; \\ L_0 = f_2^0,\ L_1 = \lambda_1 f_{2\lambda}^0 + \mu_1 f_{2\mu}^0 + \nu_1 f_{2\nu}^0, \cdots; \\ M_0 = f_3^0,\ M_1 = \lambda_1 f_{3\lambda}^0 + \mu_1 f_{3\mu}^0 + \nu_1 f_{3\nu}^0, \cdots; \\ N_0 = f_4^0,\ N_1 = \lambda_1 f_{4\lambda}^0 + \mu_1 f_{4\mu}^0 + \nu_1 f_{4\nu}^0, \cdots. \end{cases}$$

Since, according to our choice of the fundamental points, P_0 is in the present case not a fundamental point, not all of the quantities, K_0, L_0, M_0, N_0, equal zero.

Suppose $K_0 \neq 0$. We propose to prove that the cycle (C') is again a linear cycle.

For if

$$(23)\quad \left\| \begin{matrix} K_0 & L_0 & M_0 & N_0 \\ K_1 & L_1 & M_1 & N_1 \end{matrix} \right\| = 0,$$

we should have

$$(24)\quad K_1 = \rho'' K_0, \quad L_1 = \rho'' L_0, \quad M_1 = \rho'' M_0, \quad N_1 = \rho'' N_0,$$

or by applying Euler's theorem,

$$(25)\quad (\rho''\lambda_0 - \lambda_1) f_{j\lambda}^0 + (\rho''\mu_0 - \mu_1) f_{j\mu}^0 + (\rho''\nu_0 - \nu_1) f_{j\nu}^0 = 0 \qquad (j = 1, 2, 3, 4).$$

The rank * of the matrix of the coefficients in these equations is $r = 3$; hence,

$$(26)\quad \rho''\lambda_0 - \lambda_1 = 0, \quad \rho''\mu_0 - \mu_1 = 0, \quad \rho''\nu_0 - \nu_1 = 0,$$

therefore,

$$(27)\quad \left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| = 0,$$

against the assumption that (C) is a linear cycle. Therefore, we must have

$$(28)\quad \left\| \begin{matrix} K_0 & L_0 & M_0 & N_0 \\ K_1 & L_1 & M_1 & N_1 \end{matrix} \right\| \neq 0;$$

* See § 3.

that is (C') is a linear cycle of the curve K'; *hence the image* P_0' *of an* ORDINARY *point* P_0 *of* K *is an* ORDINARY POINT *of* K'.

§ 9. TRANSFORMATION OF THE MULTIPLE POINTS $A_2, \cdots, A_r$ TO THE CUBIC SURFACE.

If P_0 is one of the multiple points, $A_2, \cdots, A_r$, of order of multiplicity σ, then all points of K, in the vicinity of P_0, are represented by σ linear cycles with coinciding centers and with distinct tangents.

Let (19) and

$$(29) \qquad \begin{cases} \rho\lambda = \lambda_0 + \lambda_1' t + \lambda_2' t^2 + \cdots, \\ \rho\mu = \mu_0 + \mu_1' t + \mu_2' t^2 + \cdots, \\ \rho\nu = \nu_0 + \nu_1' t + \nu_2' t^2 + \cdots, \end{cases} \qquad (C_1)$$

be two of these cycles, then

$$(30) \qquad \left\| \begin{matrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1' & \mu_1' & \nu_1' \end{matrix} \right\| \neq 0,$$

and since the tangents to the two cycles are distinct, we must have, besides,

$$(31) \qquad \begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_1' & \mu_1' & \nu_1' \end{vmatrix} \neq 0;$$

the image (C_1') of (C_1) is given by the equations

$$(32) \qquad \begin{cases} \rho' x_1 = K_0 + K_1' t + K_2' t^2 + \cdots, \\ \rho' x_2 = L_0 + L_1' t + L_2' t^2 + \cdots, \\ \rho' x_3 = M_0 + M_1' t + M_2' t^2 + \cdots, \\ \rho' x_4 = N_0 + N_1' t + N_2' t^2 + \cdots, \end{cases} \qquad (C_1')$$

where $K_1' = \lambda_1' f_{1\lambda}^0 + \mu_1' f_{1\mu}^0 + \nu_1' f_{1\nu}^0$, etc.

Suppose (C') and (C_1') have coinciding tangents; that is

$$(33) \qquad \left\| \begin{matrix} K_0 & L_0 & M_0 & N_0 \\ K_1 & L_1 & M_1 & N_1 \\ K_1' & L_1' & M_1' & N_1' \end{matrix} \right\| = 0.$$

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Consider one of the four equations implied in (33); as

$$\begin{vmatrix} K_0 & L_0 & M_0 \\ K_1 & L_1 & M_1 \\ K_1' & L_1' & M_1' \end{vmatrix} = 0. \tag{34}$$

On account of (22) and using Euler's theorem, we obtain from (34)

$$\begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_1' & \mu_1' & \nu_1' \end{vmatrix} \cdot \begin{vmatrix} f^0_{1\lambda} & f^0_{2\lambda} & f^0_{3\lambda} \\ f^0_{1\mu} & f^0_{2\mu} & f^0_{3\mu} \\ f^0_{1\nu} & f^0_{2\nu} & f^0_{3\nu} \end{vmatrix} = 0. \tag{35}$$

But by (31)

$$\begin{vmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_1' & \mu_1' & \nu_1' \end{vmatrix} \neq 0,$$

therefore

$$\begin{vmatrix} f^0_{1\lambda} & f^0_{2\lambda} & f^0_{3\lambda} \\ f^0_{1\mu} & f^0_{2\mu} & f^0_{3\mu} \\ f^0_{1\nu} & f^0_{2\nu} & f^0_{3\nu} \end{vmatrix} = 0, \tag{36}$$

and, in the same way, it follows that all the determinants of order *three* of the matrix,

$$\| f^0_{j\lambda} \quad f^0_{j\mu} \quad f^0_{j\nu} \|, \tag{37}$$

are equal to zero; but, according to § 3, this is impossible, since also in the present case P_0 is a non-fundamental point. All the points of K' in the vicinity of P_0' are therefore represented by σ linear cycles with coinciding centers and with distinct tangents; that is, the image A_i' of one of the multiple points A_i $(i = 2, 3, \cdots, r)$ of multiplicity σ_i *is an ordinary multiple point of order* σ_i *of* K'.

§ 10. Transformation of the multiple point A_1 to the cubic surface.

If the point, P_0, is the multiple point A_1, of any order of multiplicity σ_1, then all points of K in the vicinity of P_0 are represented by σ_1 linear cycles with coinciding centers and distinct tangents.

According to our choice of the fundamental points, the point P_0 or A_1 coincides in this case with one of the fundamental points. Hence if (19) represents one of the σ_1 cycles, (C), with center P_0 and (21) its image (C') on F_3, then

$$(38) \qquad K_0 = 0, \quad L_0 = 0, \quad M_0 = 0, \quad N_0 = 0,$$

and after division by t the cycle (C') takes the form

$$(39) \qquad \begin{cases} \rho x_1 = K_1 + K_2 t + K_3 t^2 + \cdots, \\ \rho x_2 = L_1 + L_2 t + L_3 t^2 + \cdots, \\ \rho x_3 = M_1 + M_2 t + M_3 t^2 + \cdots, \\ \rho x_4 = N_1 + N_2 t + N_3 t^2 + \cdots. \end{cases} \qquad (C')$$

We propose to show that this cycle is again a *linear cycle*, therefore that

$$(40) \qquad \left\| \begin{matrix} K_1 & L_1 & M_1 & N_1 \\ K_2 & L_2 & M_2 & N_2 \end{matrix} \right\| \neq 0.$$

In order to prove it, we suppose

$$(41) \qquad \left\| \begin{matrix} K_1 & L_1 & M_1 & N_1 \\ K_2 & L_2 & M_2 & N_2 \end{matrix} \right\| = 0;$$

the quantities K_1, L_1, M_1, N_1 are not all equal to zero, as has been proved in § 4, *b*); suppose $K_1 \neq 0$, then (41) is equivalent to the three equations,

$$(42) \quad K_1 L_2 - K_2 L_1 = 0, \quad K_1 M_2 - K_2 M_1 = 0, \quad K_1 N_2 - K_2 N_1 = 0.$$

For the further discussion of these quantities, we suppose the fundamental point with which P_0 coincides to be P_3, and we make use of the special system of coördinates and the notation introduced in § 5, *d*); hence, we have

$$(43)\quad \begin{cases} K_1 = \beta_1 b_3 \mu_1, \; L_1 = -\beta_2 b_3 \lambda_1, \; M_1 = -\gamma_1 c_3 \mu_1, \; N_1 = \gamma_2 c_3 \lambda_1, \\ K_2 = \beta_1 (b_1 \lambda_1 \mu_1 + b_2 \mu_1^2 + 2 b_3 \mu_1 \nu_1 + b_3 \mu_2) - \beta_3 b_3 \lambda_1 \mu_1, \\ L_2 = -\beta_2 (b_2 \lambda_1 \mu_1 + b_1 \lambda_1^2 + 2 b_3 \lambda_1 \nu_1 + b_3 \lambda_2) + \beta_3 b_3 \lambda_1 \mu_1, \\ M_2 = -\gamma_1 (c_1 \lambda_1 \mu_1 + c_2 \mu_1^2 + 2 c_3 \mu_1 \nu_1 + c_3 \mu_2) + \gamma_3 c_3 \lambda_1 \mu_1, \\ N_2 = \gamma_2 (c_2 \lambda_1 \mu_1 + c_1 \lambda_1^2 + 2 c_3 \lambda_1 \nu_1 + c_3 \lambda_2) - \gamma_3 c_3 \lambda_1 \mu_1; \end{cases}$$

and these values substituted in (42) give

$$(44)\quad \begin{cases} \beta_1 \beta_2 (\lambda_1 \mu_2 - \lambda_2 \mu_1) + \beta_3 \lambda_1 \mu_1 (\beta_1 \mu_1 - \beta_2 \lambda_1) = 0, & \text{I}; \\ \beta_1 \gamma_1 (\alpha_1 \mu_1 - \alpha_2 \lambda_1) - a_2 \Delta b_3 c_3 \lambda_1 = 0, & \text{II}; \\ \lambda_1 \mu_1 [\beta_1 \gamma_2 (\lambda_1 \alpha_2 - \mu_1 \alpha_1) + b_3 c_3 (\beta_3 \gamma_2 \lambda_1 - \beta_1 \gamma_3 \mu_1)] \\ \qquad + \beta_1 \gamma_2 b_3 c_3 (\lambda_2 \mu_1 - \lambda_1 \mu_2) = 0, & \text{III}. \end{cases}$$

Since we supposed $K_1 \neq 0$, therefore, $\mu_1 \neq 0$.

In II, if $\lambda_1 = 0$, then $\mu_1 = 0$, against the assumption

$$(45)\quad \left\| \begin{matrix} 0 & 0 & 1 \\ \lambda_1 & \mu_1 & \nu_1 \end{matrix} \right\| \neq 0; \text{ hence, } \lambda_1 \neq 0.$$

By equating the values of $\lambda_1 \mu_2 - \lambda_2 \mu_1$ obtained from I and III, we have

$$(46)\quad \lambda_1 \alpha_2 \beta_2 \gamma_2 - \mu_1 (a_1 b_3 c_3 \Delta + \beta_2 \gamma_2 \alpha_1) = 0;$$

and by II, therefore

$$(47)\quad \lambda_1 (a_2 \Delta b_3 c_3 + \beta_1 \gamma_1 \alpha_2) - \mu_1 \alpha_1 \beta_1 \gamma_1 = 0;$$

hence

$$(48)\quad \begin{vmatrix} \alpha_2 \beta_2 \gamma_2 & a_1 b_3 c_3 \Delta + \beta_2 \gamma_2 \alpha_1 \\ a_2 b_3 c_3 \Delta + \alpha_2 \beta_1 \gamma_1 & \alpha_1 \beta_1 \gamma_1 \end{vmatrix} = 0,$$

therefore

$$(49)\quad a_1 \alpha_2 \beta_1 \gamma_1 + a_2 \alpha_1 \beta_2 \gamma_2 + a_1 a_2 b_3 c_3 \Delta = 0,$$

and finally,

$$(50)\quad \begin{vmatrix} a_1 b_1 & a_1 c_1 & b_1 c_1 \\ a_2 b_2 & a_2 c_2 & b_2 c_2 \\ a_3 b_3 & a_3 c_3 & b_3 c_3 \end{vmatrix} = 0,$$

the necessary and sufficient condition that all six fundamental points shall lie on a conic, against the hypothesis. Hence, *the cycle* (C') *of the curve* K' *is a linear cycle.*

According to § 4, the centers of the σ_1 linear cycles of the curve K' are all distinct, distributed along the fundamental line, each center being determined by the tangent line to the corresponding cycle of the curve K; and according to § 5, each center of the σ_1 linear cycles of K' does not have the center of any other linear cycle to coincide with it. Hence, *each point of the* σ_1 *points of* K' *is an* ORDINARY POINT. Therefore *the curve* K *is transformed into a new curve* K' *on the surface* F_3, *given by the equations*

$$(51)\quad \begin{cases} \rho x_1 = f_1(\lambda, \mu, \nu), \quad \rho x_2 = f_2(\lambda, \mu, \nu), \quad \rho x_3 = f_3(\lambda, \mu, \nu), \\ \rho x_4 = f_4(\lambda, \mu, \nu), \qquad\qquad \textit{combined with the equation} \\ \qquad\qquad G(\lambda, \mu, \nu) = 0, \end{cases}$$

which has the multiple point A_1 *of* K *resolved into ordinary points and all the remaining multiple points* $A_2, \ldots, A_r$ *of* K *transformed into new multiple points* $A_2, \cdots, A_r'$ *unchanged in orders of multiplicity, with linear cycles and with distinct tangents, and without the introduction of any new multiple points or other singularities.*

CHAPTER IV.

Transformation from the Surface F_3 to a Plane Π'.

§ 11. One-to-two correspondence between the surface F_3 and the plane Π'.

We now select a point O on F_3, and a plane Π' not passing through O, and project K' from O to Π', thus obtaining a plane algebraic curve K''. We are going to prove, that the point O can be so chosen, that the correspondence thus established between the curves, K' and K'', is a one-to-one correspondence of points and that the curve K'' has $r-1$ ordinary multiple points, $A_2'', \cdots, A_r''$, of the same multiplicities as the points $A_2, \cdots, A_r$ of K, and besides no other multiple points, except ordinary double points.

Formulae for the correspondence between F_3 and Π':

We introduce a new tetrahedron of reference, one of whose vertices, $\xi = 0, \eta = 0, \zeta = 0, \tau = 1$, coincides with the point O, while the opposite face ABC, $\tau = 0$, coincides with the plane Π'. If $\xi_0, \eta_0, \zeta_0, \tau_0$ are the coördinates of any point, P_0', on F_3 different from O, the projecting ray, OP_0', is given in parameter representation by the equations,

$$(1) \qquad \rho\xi = t_1\xi_0, \qquad \rho\eta = t_1\eta_0, \qquad \rho\zeta = t_1\zeta_0, \qquad \rho\tau = t_1\tau_0 + t_2.$$

The coördinates of the projection, P_0'', are

$$(2) \qquad \rho\xi = \xi_0, \qquad \rho\eta = \eta_0, \qquad \rho\zeta = \zeta_0, \qquad \tau = 0.$$

The quantities ξ_0, η_0, ζ_0, which are not all three zero, are at the same time the trilinear coördinates * of the point, P_0'', with respect to the triangle of reference ABC in Π'. Only when the point, P_0', coincides with O does its projection become indeterminate. In order to avoid this exceptional case in the projection of K', we impose, upon the point O, *condition I*: *that the point O shall not lie on the curve K'*.

* Clebsch-Lindemann "Vorlesungen über Geometrie," Bd. II, p. 99.

Vice versa: If ξ_0, η_0, ζ_0 are the trilinear coördinates of any point P_0'' in Π' with respect to the triangle of reference ABC, then the tetrahedral coördinates of the point P_0'' are ξ_0, η_0, ζ_0, 0, and the projecting ray OP_0'' is given by the equations,

$$(3) \qquad \rho\xi = \xi_0 t_1, \qquad \rho\eta = \eta_0 t_1, \qquad \rho\zeta = \zeta_0 t_1, \qquad \rho\tau = t_2.$$

If we substitute these values in the equation of the cubic surface, we will get a cubic equation in $t_1 : t_2$. If this cubic equation does not degenerate into an identity its three roots will give the parameters of the three points of intersection of the ray OP'' with F_3. One of these points is O, the two other points we denote by P_1' and P_2'. If the cubic equation degenerates into an identity, it is satisfied for all values of $t_1 : t_2$, and therefore the ray lies wholly on F_3. In order to avoid the exceptional case, we impose, upon the point O, *condition II*: *that the point O shall not lie on any one of the twenty-seven* right lines on F_3*. We thus obtain a one-to-two correspondence of points between F_3 and Π'; the only exception is the point O on F_3, to which corresponds in the plane Π' the line of intersection of the tangent plane† to F_3 at O with the plane Π'. The locus made up of the twenty-seven right lines on F_3, we denote by Λ_1.

§ 12. Related points.

If λ_1^0, μ_1^0, ν_1^0 and λ_2^0, μ_2^0, ν_2^0 are the parameters of the two points P_1' and P_2' on F_3, and if P_1' does not coincide with O and does not lie on one of the six fundamental lines, then $\lambda_2^0 : \mu_2^0 : \nu_2^0$ are rationally expressible in terms of λ_1^0, μ_1^0, ν_1^0.

The coördinates of P_1', in the new system, are

$$(4) \qquad \begin{cases} \xi_1^0 = f_1'(\lambda_1^0, \mu_1^0, \nu_1^0), & \eta_1^0 = f_2'(\lambda_1^0, \mu_1^0, \nu_1^0), \\ \zeta_1^0 = f_3'(\lambda_1^0, \mu_1^0, \nu_1^0), & \tau_1^0 = f_4'(\lambda_1^0, \mu_1^0, \nu_1^0), \end{cases}$$

where f_1', f_2', f_3', f_4' are the same linear combinations of f_1, f_2, f_3,

* See § 6.

† There exists a definite tangent plane to F_3 at O, since O is not a singular point of F_3 (see § 5).

f_4 as ξ, η, ζ, τ are of x_1, x_2, x_3, x_4; hence the equations for the projecting ray OP'_1 become

$$(5)\quad \begin{cases} \rho\xi = t_1 f'_1(\lambda_1^0, \mu_1^0, \nu_1^0), & \rho\eta = t_1 f'_2(\lambda_1^0, \mu_1^0, \nu_1^0), \\ \rho\zeta = t_1 f'_3(\lambda_1^0, \mu_1^0, \nu_1^0), & \rho\tau = t_1 f'_4(\lambda_1^0, \mu_1^0, \nu_1^0) + t_2. \end{cases}$$

If λ, μ, ν are the parameters of one of the points P' of intersection of OP'_1 with F_3, then the coördinates of P' are

$$(6)\quad \begin{cases} \xi = t_3 f'_1(\lambda, \mu, \nu), & \eta = t_3 f'_2(\lambda, \mu, \nu), \\ \zeta = t_3 f'_3(\lambda, \mu, \nu), & \tau = t_3 f'_4(\lambda, \mu, \nu), \quad t_3 \neq 0. \end{cases}$$

Since P' lies at the same time on OP'_1 it must be possible to so determine $t_1 : t_2 : t_3$ that

$$(7)\quad \begin{cases} t_1 f'_1(\lambda_1^0, \mu_1^0, \nu_1^0) = t_3 f'_1(\lambda, \mu, \nu), \\ t_1 f'_2(\lambda_1^0, \mu_1^0, \nu_1^0) = t_3 f'_2(\lambda, \mu, \nu), \\ t_1 f'_3(\lambda_1^0, \mu_1^0, \nu_1^0) = t_3 f'_3(\lambda, \mu, \nu), \\ t_1 f'_4(\lambda_1^0, \mu_1^0, \nu_1^0) + t_2 = t_3 f'_4(\lambda, \mu, \nu). \end{cases}$$

To obtain the points of intersection, (7) must be solved with respect to $\lambda : \mu : \nu$ and $t_1 : t_2 : t_3$.

From (7) we get

$$(8)\quad \left\| \begin{matrix} f'_1(\lambda_1^0, \mu_1^0, \nu_1^0) & f'_2(\lambda_1^0, \mu_1^0, \nu_1^0) & f'_3(\lambda_1^0, \mu_1^0, \nu_1^0) \\ f'_1(\lambda, \mu, \nu) & f'_2(\lambda, \mu, \nu) & f'_3(\lambda, \mu, \nu) \end{matrix} \right\| = 0,$$

not all of the quantities,

$$f'_1(\lambda_1^0, \mu_1^0, \nu_1^0), f'_2(\lambda_1^0, \mu_1^0, \nu_1^0), f'_3(\lambda_1^0, \mu_1^0, \nu_1^0),$$

are equal to zero, since P'_1 is different from O and does not lie on one of the six fundamental lines.

Set $f'_1(\lambda_1^0, \mu_1^0, \nu_1^0) \neq 0$; then (8) is equivalent to the two equations

$$(9)\quad \begin{cases} f'_1(\lambda_1^0, \mu_1^0, \nu_1^0) \cdot f'_2(\lambda, \mu, \nu) - f'_1(\lambda, \mu, \nu) \cdot f'_2(\lambda_1^0, \mu_1^0, \nu_1^0) = 0, \\ f'_1(\lambda_1^0, \mu_1^0, \nu_1^0) \cdot f'_3(\lambda, \mu, \nu) - f'_1(\lambda, \mu, \nu) \cdot f'_3(\lambda_1^0, \mu_1^0, \nu_1^0) = 0. \end{cases}$$

These two cubics can not contain any common factor, because OP'_1 can

not lie wholly on F_3, since O does not lie on one of the twenty-seven straight lines of F_3, they intersect therefore in nine points; eight of these points are the points, $P_1, \cdots, P_6, (\lambda_1^0, \mu_1^0, \nu_1^0)$, and a point $(\lambda^0, \mu^0, \nu^0)$, which is the image of O in Π. The ninth base point $P_2^0(\lambda_2^0, \mu_2^0, \nu_2^0)$ is uniquely determined, and therefore $\lambda_2^0 : \mu_2^0 : \nu_2^0$ are rational functions of $\lambda_1^0 : \mu_1^0 : \nu_1^0$; say, *

$$(10)\quad \rho\lambda_2^0 = \phi(\lambda_1^0, \mu_1^0, \nu_1^0),\ \rho\mu_2^0 = \psi(\lambda_1^0, \mu_1^0, \nu_1^0),\ \rho\nu_2^0 = \chi(\lambda_1^0, \mu_1^0, \nu_1^0),$$

where ϕ, ψ, χ are homogeneous integral functions of λ_1^0, μ_1^0, ν_1^0.

§ 13. One-to-one correspondence between the curves K and K''.

Although the projection from the point O establishes a one-to-two correspondence between the surface F_3 and the plane Π', nevertheless the correspondence between the two curves K' and K'' defined by the same projection is a one-to-one correspondence provided the curve K in the plane Π is *irreducible*, as we have supposed.

In order to prove it, we consider any point P_1' of K' and ask under what conditions will the related point P_2' to P_1' also lie on K'?

The curve K', in the new system of coördinates, is given by the equations

$$(11)\quad \begin{cases} \rho\xi = f_1'(\lambda, \mu, \nu),\ \rho\eta = f_2'(\lambda, \mu, \nu),\ \rho\zeta = f_3'(\lambda, \mu, \nu), \\ \rho\tau = f_4'(\lambda, \mu, \nu),\ G(\lambda, \mu, \nu) = 0. \end{cases}$$

If P_1' and P_2' both lie on K', then we have

$$(12)\qquad G(\lambda_1^0, \mu_1^0, \nu_1^0) = 0,\quad G(\lambda_2^0, \mu_2^0, \nu_2^0) = 0;$$

the latter equation according to § 12, may be written

$$(13)\quad G[\phi(\lambda_1^0, \mu_1^0, \nu_1^0),\ \psi(\lambda_1^0, \mu_1^0, \nu_1^0),\ \chi(\lambda_1^0, \mu_1^0, \nu_1^0)] \equiv G_1(\lambda_1^0, \mu_1^0, \nu_1^0) = 0.$$

* By an easy limiting process it can be shown that the formulæ (10) still hold for the excluded points, with the understanding that the ratios $\varphi : \psi : \chi$ are replaced by their limits as the point $(\lambda_1^0, \mu_1^0, \nu_1^0)$ approaches, along a given path, one of the excluded points.

Hence the coördinates must satisfy simultaneously the two equations

$$G(\lambda_1^0, \mu_1^0, \nu_1^0) = 0, \quad G_1(\lambda_1^0, \mu_1^0, \nu_1^0) = 0, \tag{14}$$

if P_2' lies on K'.

The equation, $G_1(\lambda_1^0, \mu_1^0, \nu_1^0) = 0$, can not reduce to an identity holding for all values of $\lambda_1^0, \mu_1^0, \nu_1^0$; for, let Q' be any point on F_3, different from O and not on K', if we choose for P_1', the third point of intersection of OQ' with F_3, then the point P_2' coincides with Q' and therefore does not lie on K', and hence $G_1(\lambda_1^0, \mu_1^0, \nu_1^0) \neq 0$.

We now make use of the hypothesis that K is *irreducible* *; then the two equations

$$G(\lambda, \mu, \nu) = 0, \quad G_1(\lambda, \mu, \nu) = 0, \tag{15}$$

have either no common factor, or the factor $G(\lambda, \mu, \nu)$. In case first, G and G_1 can intersect in only a *finite number* of points; hence, there can be only a *finite number* of points P_1' on K' for which P_2' falls on K'. In case second, for *all points* P_1' on K' the *related points* P_2' lie also on K'.

Hence, if we can find *one* particular position of P_1' for which P_2' *does not* lie on K', then we have case first.

Now, the tangent plane to F_3 at O meets K' in at least one point, say P_1', and the ray OP_1' meets F_3 in the three points, P_1', O, O; therefore the point P_2' must be the point O and hence does not fall on the curve K', since O † is not on K'. There exists therefore only a finite number m of pairs of related points, P_1', P_2', which lie both on K'; the projections of two such related points coincide in a point of the curve K'' and give rise to a new multiple point of K''. These m new multiple points will be denoted by $D_1'', \cdots, D_m''$. To every other point of K'' corresponds one, and but one, point of K', and in this sense, *the projection from the point O establishes indeed a one-to-one correspondence between the two curves K' and K''*.

* See § 8.

† See § 11.

§ 14. THE ORDERS OF THE CYCLES OF THE CURVE K''.

Let

$$(16)\qquad \begin{cases} \rho\xi = \xi_0 + \xi_1 t + \xi_2 t^2 + \cdots, \\ \rho\eta = \eta_0 + \eta_1 t + \eta_2 t^2 + \cdots, \\ \rho\zeta = \zeta_0 + \zeta_1 t + \zeta_2 t^2 + \cdots, \\ \rho\tau = \tau_0 + \tau_1 t + \tau_2 t^2 + \cdots, \end{cases} \qquad (C_0')$$

be a cycle of the curve K' expressed in the new system of coördinates. Since, according to §§ 8, 9, 10, all cycles of K' are linear, we have

$$(17)\qquad \left\| \begin{matrix} \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \end{matrix} \right\| \neq 0,$$

where not all three of the quantities, ξ_0, η_0, ζ_0, are equal to zero, since the center of (C_0') is different from the point $O\,(0, 0, 0, 1)$, which is not on K'.

The cycle (C_0') is projected into a new cycle (C_0'') in Π', given by the equations

$$(18)\qquad \begin{cases} \rho\xi = \xi_0 + \xi_1 t + \xi_2 t^2 + \cdots, \\ \rho\eta = \eta_0 + \eta_1 t + \eta_2 t^2 + \cdots, \\ \rho\zeta = \zeta_0 + \zeta_1 t + \zeta_2 t^2 + \cdots. \end{cases}$$

We ask under what conditions will (C_0'') be of higher order than the first?

This will be the case if, and only if,

$$(19)\qquad \left\| \begin{matrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \end{matrix} \right\| = 0.$$

But if (19) is satisfied, we can determine two quantities k_1, k_2, so that

$$(20)\qquad 0 = \xi_0 k_1 + \xi_1 k_2, \quad 0 = \eta_0 k_1 + \eta_1 k_2, \quad 0 = \zeta_0 k_1 + \zeta_1 k_2,$$

and therefore the point O lies on the tangent to the cycle (C_0'),

which may be written *

$$(21)\qquad \begin{cases} \rho\xi = \xi_0 t_1 + \xi_1 t_2, & \rho\eta = \eta_0 t_1 + \eta_1 t_2, \\ \rho\zeta = \zeta_0 t_1 + \zeta_1 t_2, & \rho\tau = \tau_0 t_1 + \tau_1 t_2. \end{cases}$$

If, therefore, we construct, in every point P' of K', the tangent to K', or, in case P' should be a multiple point of K', the tangents to the linear cycles with coinciding centers at P' and denote by Q', in each case, the third point of intersection † of the tangent with F_3, then as the point P' describes the curve K' the point Q' will describe a curve on F_3, which we denote by Λ_2. If now we impose *condition III* upon the center of projection O, *that the point O shall be external to this locus* Λ_2, then linear cycles of K' are projected into linear cycles of K'', and since all the cycles of K' are linear, also *all the cycles of* K'' *will be linear.*

§ 15. The projection of the multiple points $A'_2, \cdots, A'_r$ of K'.

Let (16) and

$$(22)\qquad \begin{cases} \rho\xi = \xi_0 + \xi'_1 t + \xi'_2 t^2 + \cdots, \\ \rho\eta = \eta_0 + \eta'_1 t + \eta'_2 t^2 + \cdots, \\ \rho\zeta = \zeta_0 + \zeta'_1 t + \zeta'_2 t^2 + \cdots, \\ \rho\tau = \tau_0 + \tau'_1 t + \tau'_2 t^2 + \cdots, \end{cases} \qquad (C'_1)$$

be two of the linear cycles with the multiple point A'_i of K' for center, then

$$(23)\qquad \begin{Vmatrix} \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi'_1 & \eta'_1 & \zeta'_1 & \tau'_1 \end{Vmatrix} \neq 0;$$

and according to § 9 (C'_0) and (C'_1) have distinct tangents, that is

$$(24)\qquad \begin{Vmatrix} \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi'_1 & \eta'_1 & \zeta'_1 & \tau'_1 \end{Vmatrix} \neq 0.$$

* See § 7.

† It might happen that the tangent lies wholly on the surface F_3, in which case it would coincide with one of the twenty-seven straight lines of F_3. But since we have already imposed upon O the condition not to lie on one of these twenty-seven straight lines, no special provision is necessary for this contingency.

The new cycle (C_1'') of K'' is given by the equations

$$(25)\qquad \begin{cases} \rho\xi = \xi_0 + \xi_1' t + \xi_2' t^2 + \cdots, \\ \rho\eta = \eta_0 + \eta_1' t + \eta_2' t^2 + \cdots, \\ \rho\zeta = \zeta_0 + \zeta_1' t + \zeta_2' t^2 + \cdots. \end{cases}$$

The two cycles (C_0'') and (C_1'') will have coinciding tangents if, and only if,

$$(26)\qquad \begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_1' & \eta_1' & \zeta_1' \end{vmatrix} = 0.$$

This equation has a simple geometrical meaning: the equation of the plane passing through the tangents to (C_0') and (C_1') is, according to § 7,

$$(27)\qquad \begin{vmatrix} \xi & \eta & \zeta & \tau \\ \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_1' & \eta_1' & \zeta_1' & \tau_1' \end{vmatrix} = 0;$$

now if the point O lies in the plane (27), we have

$$(28)\qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_1' & \eta_1' & \zeta_1' & \tau_1' \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_1' & \eta_1' & \zeta_1' \end{vmatrix} = 0.$$

This plane (27) is the tangent plane to F_3 at P_0'; for the tangents to the cycles (C_0') and (C_1') are, at the same time, tangents to the surface F_3; and since F_3 has a definite tangent plane at A_i' (see § 5),

this tangent plane must be identical with the plane through the two distinct tangents to the cycles (C_0') and (C_1').

The plane (27) cuts the surface F_3 in a curve H. Hence, if we construct tangent planes to F_3 at each multiple point $A_2', \cdots, A_r'$ of K', we shall obtain $r-1$ curves $H_2, \cdots, H_r$, which constitute together a locus which we denote by Λ_3; and if we impose, upon the center of projection O, *condition IV, that O shall be external to this locus* Λ_3, then for each one of the multiple points $A_2', \cdots, A_r'$ of K' *the σ_i linear cycles of A_i' will be projected into σ_i linear cycles with the same center A_i'' and with distinct tangents.*

From the conditions I—IV imposed so far upon the point O, it follows therefore that if P'' be any point of K'' different from the m points $D_1'', \cdots, D_m''$, then *either* P'' is the projection of an ordinary point of K', in which case P'' is again an ordinary point of K''; or P'' is the projection of one of the multiple points A_i' $(i = 2, \cdots, r)$, with σ_i linear cycles with distinct tangents, in which case P'' is an ordinary multiple point of K'' with the same number σ_i of linear cycles and with distinct tangents.

It remains therefore only to study the points $D_1'', \cdots, D_m''$ of K''.

§ 16. Condition to prevent the points D'' from being of higher multiplicity than the second order.

In each point D'' coincide the projections of two related points P_1', P_2' of K'. If the two points P_1', P_2' are ordinary points of K', the point D'' will be a double point of K''; if one of them is one of the multiple points A_i', the point D'' will be a multiple point of K'' of higher multiplicity.

The latter case will present itself if, and only if, the third point of intersection of the line OA_i' with the surface F_3 lies on the curve K'. Hence the projection of A_i' can not coincide with the image of any other point P_0' of K', if the point O does not lie on the curve Γ_i described by the third * point of intersection Q' of the line $A_i'P_0'$

* For particular positions of the point P_0' the line $A_i'P_0'$ might lie wholly on the surface F_3; the center O will then certainly not lie on the line $A_i'P_0'$, since we have supposed that O shall not lie on one of the straight lines of the surface F_3. See § 11.

with F_3 as the point P'_0 describes K'. If we denote by Λ_4 the locus made up of the $r-1$ curves $\Gamma_2, \Gamma_3, \cdots, \Gamma_r$ and impose *condition V* upon the point O, *that O shall be external to this locus* Λ_4, then the image of none of the multiple points $A'_2, \cdots, A'_r$ will coincide with the image of any other point, simple or multiple, of the curve K'. *The points* $D''_1, \cdots, D''_m$ *will therefore be double points of* K''; and on the other hand, if A'_i is the center of σ_i linear cycles of the curve K', then the image A''_i of A'_i, in the plane Π', will likewise be the center of σ_i (and not more) linear cycles.

§ 17. Condition to prevent coinciding tangents at the double points $D''_1, \cdots, D''_m$.

Let D'' be any one of the double points $D''_1 \cdots, D''_m$ and $P'_1(\xi_1, \eta_1, \zeta_1, \tau_1)$ and $P'_2(\xi_2, \eta_2, \zeta_2, \tau_2)$ the two related points of F_3, both lying on K', whose projections $P''_1(\xi_1, \eta_1, \zeta_1)$ and $P''_2(\xi_2, \eta_2, \zeta_2)$ in the plane Π' coincide in the point D'', so that

$$\left\| \begin{matrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{matrix} \right\| = 0, \tag{29}$$

at the same time not all three of the quantities ξ_1, η_1, ζ_1, nor ξ_2, η_2, ζ_2 can be equal to zero, since on account of condition I neither P'_1 nor P'_2 coincides with O. Further the points P'_1 and P'_2 are distinct, according to § 13, hence

$$\left\| \begin{matrix} \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \end{matrix} \right\| \neq 0. \tag{30}$$

According to § 16, P'_1 and P'_2 are both ordinary points of K'; let

$$\left\{ \begin{aligned} \rho\xi &= \xi_1 + \xi'_1 t + \cdots, \\ \rho\eta &= \eta_1 + \eta'_1 t + \cdots, \\ \rho\zeta &= \zeta_1 + \zeta'_1 t + \cdots, \\ \rho\tau &= \tau_1 + \tau'_1 t + \cdots, \end{aligned} \right. \tag{31} \qquad (C'_1)$$

and

$$(32)\qquad \left\{\begin{aligned} \rho\xi &= \xi_2 + \xi_2' t + \cdots, \\ \rho\eta &= \eta_2 + \eta_2' t + \cdots, \\ \rho\zeta &= \zeta_2 + \zeta_2' t + \cdots, \\ \rho\tau &= \tau_2 + \tau_2' t + \cdots, \end{aligned}\right. \qquad (C_2')$$

be the two linear cycles which represent the curve K' in the vicinity of P_1' and P_2' respectively.

The projections, in the plane Π', of these cycles are

$$(33)\qquad \left\{\begin{aligned} \rho\xi &= \xi_1 + \xi_1' t + \cdots, \\ \rho\eta &= \eta_1 + \eta_1' t + \cdots, \\ \rho\zeta &= \zeta_1 + \zeta_1' t + \cdots, \end{aligned}\right. \qquad (C_1'')$$

and

$$(34)\qquad \left\{\begin{aligned} \rho\xi &= \xi_2 + \xi_2' t + \cdots, \\ \rho\eta &= \eta_2 + \eta_2' t + \cdots, \\ \rho\zeta &= \zeta_2 + \zeta_2' t + \cdots, \end{aligned}\right. \qquad (C_2'')$$

respectively; their coinciding center is the point $\xi_1 : \eta_1 : \zeta_1 = \xi_2 : \eta_2 : \zeta_2$.

The tangents to the two cycles (C_1'') and (C_2'') will coincide if, and only if,

$$(35)\qquad \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_1' & \eta_1' & \zeta_1' \\ \xi_2' & \eta_2' & \zeta_2' \end{vmatrix} = 0.$$

This equation has a simple geometrical meaning: the equation of the plane through the tangent to the cycle (C_1') and the point P_2' is

$$(36)\qquad \begin{vmatrix} \xi & \eta & \zeta & \tau \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_1' & \eta_1' & \zeta_1' & \tau_1' \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \end{vmatrix} = 0,$$

the condition that this plane (which on account of (29) always passes

through O) contains the tangent to the cycle (C'_2) is

$$\begin{vmatrix} \xi'_2 & \eta'_2 & \zeta'_2 & \tau'_2 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \zeta'_1 & \eta'_1 & \zeta'_1 & \tau'_1 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \end{vmatrix} = 0, \tag{37}$$

since the point $\xi'_2, \eta'_2, \zeta'_2, \tau'_2$ is a point of the tangent to (C'_2).

But according to (29), a quantity $\rho \neq 0$ can be determined so that

$$\xi_2 = \rho\xi_1, \qquad \eta_2 = \rho\eta_1, \qquad \zeta_2 = \rho\zeta_1. \tag{38}$$

Hence if we subtract, in the determinant (37), from the last row the second multiplied by ρ and notice that $\tau_2 \neq \rho\tau_1$ on account of (30), we obtain (35).

The tangents to the two cycles $(C''_1), (C''_2)$ with the center D'' will therefore coincide if, and only if, the tangents to the two cycles (C'_1) and (C'_2) lie in one plane passing through O.

We must now impose upon the point O, a condition which will prevent the point O from lying in a plane containing two tangents to K'. For this purpose, let P'_1 be any ordinary point of K', and let

$$\alpha\bar{\xi} + \beta\bar{\eta} = 0 \tag{39}$$

represent the pencil of planes through the tangent to K' at P'_1; $\bar{\xi}$, $\bar{\eta}$, being homogeneous linear functions of ξ, η, ζ, τ.

Let P'_2 be another point of K', the image of an ordinary point P_2 of K; let

$$\begin{cases} \lambda = \lambda_0 + \lambda_1 t + \cdots, \\ \mu = \mu_0 + \mu_1 t + \cdots, \\ \nu = \nu_0 + \nu_1 t + \cdots, \end{cases} \tag{40}$$

be the linear cycle which represents the curve K in the vicinity of P_2; then the series (40) satisfy identically the equation of the curve K,

$$G(\lambda, \mu, \nu) = 0, \tag{41}$$

hence

$$G(\lambda_0, \mu_0, \nu_0) = 0, \tag{42}$$

$$\lambda_1 G_\lambda(\lambda_0, \mu_0, \nu_0) + \mu_1 G_\mu(\lambda_0, \mu_0, \nu_0) + \nu_1 G_\nu(\lambda_0, \mu_0, \nu_0) = 0. \tag{43}$$

The tangent to K' at P_2' is, according to § 8, given by the equations

$$\begin{cases} \rho\xi = f_1^0 + s_1^0 t, & \rho\eta = f_2^0 + s_2^0 t, \\ \rho\zeta = f_3^0 + s_3^0 t, & \rho\tau = f_4^0 + s_4^0 t, \end{cases} \tag{44}$$

where

$$\begin{cases} f_j^0 = f_j(\lambda_0, \mu_0, \nu_0) & (j = 1, 2, 3, 4), \\ s_j^0 = \lambda_1 f_{j\lambda}(\lambda_0, \mu_0, \nu_0) + \mu_1 f_{j\mu}(\lambda_0, \mu_0, \nu_0) + \nu_1 f_{j\nu}(\lambda_0, \mu_0, \nu_0). \end{cases} \tag{45}$$

Hence if the tangent to K' at P_2' is to lie in the plane (39), it is necessary and sufficient that

$$\begin{cases} \alpha\bar{f}_1^0 + \beta\bar{f}_2^0 = 0, \\ \alpha\bar{s}_1^0 + \beta\bar{s}_2^0 = 0, \end{cases} \tag{46}$$

where $\bar{f}_1^0, \bar{f}_2^0$, $(\bar{s}_1^0, \bar{s}_2^0)$ are the same linear functions of $f_1^0, \cdots, f_4^0$, $(s_1^0, \cdots, s_4^0)$ as ξ, η are of ξ, η, ζ, τ. Hence we must have

$$\bar{f}_1^0 \bar{s}_2^0 - \bar{f}_2^0 \bar{s}_1^0 = 0, \tag{47}$$

or, if we make use of EULER'S theorem,

$$\begin{aligned} (\mu_0\nu_1 - \mu_1\nu_0)(\bar{f}_{1\mu}^0 \bar{f}_{2\nu}^0 - \bar{f}_{2\mu}^0 \bar{f}_{1\nu}^0) + (\lambda_0\mu_1 - \mu_0\lambda_1)(\bar{f}_{1\lambda}^0 \bar{f}_{2\mu}^0 - \bar{f}_{2\lambda}^0 \bar{f}_{1\mu}^0) \\ + (\lambda_0\nu_1 - \lambda_1\nu_0)(\bar{f}_{1\lambda}^0 \bar{f}_{2\nu}^0 - \bar{f}_{2\lambda}^0 \bar{f}_{1\nu}^0) = 0. \end{aligned} \tag{48}$$

But the equation (42) may be written

$$\lambda_0 G_\lambda(\lambda_0, \mu_0, \nu_0) + \mu_0 G_\mu(\lambda_0, \mu_0, \nu_0) + \nu_0 G_\nu(\lambda_0, \mu_0, \nu_0) = 0, \tag{49}$$

therefore

$$\begin{cases} \mu_0\nu_1 - \mu_1\nu_0 = \sigma G_\lambda(\lambda_0, \mu_0, \nu_0), \\ \lambda_1\nu_0 - \lambda_0\nu_1 = \sigma G_\mu(\lambda_0, \mu_0, \nu_0), \\ \lambda_0\mu_1 - \lambda_1\mu_0 = \sigma G_\nu(\lambda_0, \mu_0, \nu_0). \end{cases} \tag{50}$$

Hence we obtain the result: If the tangent to K' at P_2' lies in one of the planes of the pencil (39) then the parameters λ_0, μ_0, ν_0 must satisfy the two equations

$$H(\lambda, \mu, \nu) \equiv \begin{vmatrix} \bar{f}_{1\lambda} & \bar{f}_{2\lambda} & G_\lambda \\ \bar{f}_{1\mu} & \bar{f}_{2\mu} & G_\mu \\ \bar{f}_{1\nu} & \bar{f}_{2\nu} & G_\nu \end{vmatrix} = 0, \tag{51}$$

and

$$G(\lambda, \mu, \nu) = 0. \tag{52}$$

Vice versa: If λ_0, μ_0, ν_0 satisfy these equations there exists a plane of the pencil (39), which contains the tangent to K' at P_2'.

Since (52) is *irreducible* the two curves (51) and (52) have either a finite number of points of intersection, or else H is divisible by G, in which case *every* tangent to K' is contained in a plane of the pencil (39).

In the latter case, the curve K' would be a plane curve. To prove it, we use for greater convenience non-homogeneous rectangular coördinates x, y, z, the z-axis coinciding with the tangent to K' at P_1', and origin with the point P_1'. If the curve K' is given in parameter representation by

$$x = \phi(t), \qquad y = \psi(t), \qquad z = \chi(t), \tag{53}$$

and if the tangent at the point t

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}, \tag{54}$$

is to pass through the z-axis, we must have

$$xy' - yx' = 0, \tag{55}$$

and if this is true for every tangent, then we obtain, by integrating (55),

$$y = cx, \tag{56}$$

where c is a constant. That is, the curve K' lies in a plane passing through the z-axis.

If the curve K' is a plane curve, the curve K is a curve of the system of cubics through the six fundamental points. But we have supposed that K is of higher order than the third (see § 8), hence this exceptional case can not take place, and there exists therefore only a finite number of points P_2' on K' in which the tangent to K' lies in a plane of the pencil (39). For each such point P_2' draw the line $P_1' P_2'$, it meets the cubic surface F_3 in a third point R. As the point P_1' describes the curve K', these points R will describe a finite number of curves on F_3. The totality of these curves makes up a locus which we denote by Λ_5. Hence if we impose upon the center of projection O, *condition VI, that* O *shall not lie on this locus* Λ_5, then *the new double points* $D_1'', \cdots, D_m''$ *will have distinct tangents.*

We will now give a résumé of our results. There was given in the plane Π an irreducible algebraic curve K (of order higher than the third) with r ordinary multiple points $A_1, A_2, \cdots, A_r$. We selected the six fundamental points of the transformation, so that no three lie on a straight line and not all six on a conic; we chose one of them at the point A_1, the other five external to the curve K. We thus obtained a one-to-one correspondence between the curve K and its image K' on the surface F_3. This curve K', we projected from a point O of the surface F_3 upon a second plane Π'. We imposed upon the point O the conditions not to lie on certain curves, viz., K' and the loci denoted by $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$.

Then we obtained a one-to-one correspondence between the curve K' and its image K'' in the plane Π', and the curve K'' has, corresponding to the multiple points $A_2, \cdots, A_r$, $r-1$ ordinary multiple points $A_2'', \cdots, A_r''$ of the same respective multiplicities as $A_2, \cdots, A_r$; whereas, the multiple point A_1 has been resolved into σ_1 ordinary points. Besides the curve K'' has a certain number of new double points $D_1'', \cdots, D_m''$ with linear cycles and distinct tangents.

The points $A_2'', \cdots, A_r''$; $D_1'', \cdots, D_m''$ are the only singular points of K''.

We now apply to the curve K'' and to the multiple point A_2'' the same process which has been applied to the curve K and the multiple point A_1, and so on. We repeat the process as many times as there

are multiple points $A_1, \cdots, A_r$ to be resolved, taking each time the new curve as the curve to be transformed, and finally a curve will be reached possessing a finite number of ordinary double points with distinct tangents, but no other multiple points.

Combining this result with NOETHER'S Theorem mentioned in the introduction, we have the final theorem that *every algebraic curve can be transformed, by a birational transformation of the curve, into a curve which has no other multiple points but ordinary double points.*

AUTOBIOGRAPHY

I WAS born on the 20th of August, 1863, in Oktibbeha County, Mississippi. I attended the public schools of this county for four years, the High School for four years, and the Mississippi Agricultural and Mechanical College for three years, graduating in 1883 with the degree of B.S., and in 1886 received the degree of M.S. I attended the University of Virginia, the summers of 1885–86–87; the Universities of Göttingen and Berlin the semesters of 1888–89. In 1895 I entered the University of Chicago as a student of mathematics and astronomy, and studied twelve quarters. I was assistant in mathematics in the Mississippi A. & M. College from 1883–1888, Professor and Head of the department from 1888–, and Director of the School of Engineering from 1902–. I received instruction in mathematics from Professors Thornton, Schoenflies, Schwarz, Hensel, Fuchs, Moore, Bolza, and Maschke, and in astronomy from Professors Laves and Moulton. My thesis was carried on with Professor Bolza. I feel under a deep obligation to all the Professors named, but especially to Professor Bolza for the continued and varied assistance given me throughout my whole time of graduate work.

BUZ M. WALKER.

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